1. **Counting**

   - How many ways can you scramble the word Aardvark?

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2. **Bayes Rule and Conditional Prob Burnt pancakes**
   I have a bag containing three pancakes: One golden on both sides, one burnt on both sides, and one golden on one side and burnt on the other. You shake the bag, draw a pancake at random, look at one side, and notice that it is burnt. What is the probability that the other side is burnt? Show your work.

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3. **Random Variables** Given two independent geometric RVs, $X_1$, $X_2$. What is the distribution of the $Y = \min(X_1, X_2)$? If $Z = \max(X_1, X_2)$, what is $P(Z = n|Y = 3)$?

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4. **Expectation** Each pixel in a $32 \times 8$ display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let $X$ denote the number of horizontal lines that the display shows.

   - What is $E[X]$?
   - What is $Var[X]$?
   - Show that $Pr[X \geq 2] \leq 1/25$.

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5. **Continuous Distributions** Alice and Bob agree to meet in the park at noon. Alice lateness is exponentially distributed with mean 30 minutes and Bob's lateness is Uniformly distributed with mean 30 minutes. If their arrival times are independent, what is the probability they arrive within ten minutes of each other?

6. **Continuous Expectations:** Say you break a stick of length $L$ in two places uniformly at random. What is the expected distance between the breaks?

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7. **Infinity and Countability:** Consider the set of all binary strings (of both finite and infinite length).

   - Is this set: Countable or Uncountable?
   - If we remove all strings with a particular prefix has the cardinality changed?
   - if we remove all strings of infinite length, has the cardinality changed?

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8. **Counting Degrees** In a graph, the degree of a vertex is the number of edges incident to that vertex. Prove that the sum of the degrees of all the vertices is always an even number.

9. **Trees.** Prove the equivalence of the following definitions for a graph $G$ being a tree:

   (a) $G$ is a connected graph with no cycles

   (b) $G$ is a connected graph on $n$ vertices with $n - 1$ edges

(c) $G$ is a graph on $n$ vertices with $n-1$ edges with no cycles.

(d) $G$ is a graph such that, if you add any edge, the resulting graph has a unique cycle

(e) $G$ is a connected graph such that, if you delete any edge it becomes disconnected.

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10. **Interpolation practice**
Find a polynomial $h(x) = ax^2 + bx + c$ of degree at most 2 such that $h(0) \equiv 3 \pmod 7$, $h(1) \equiv 6 \pmod 7$, and $h(2) \equiv 6 \pmod 7$.

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11. **[Fibonacci numbers and modular arithmetic]**
Recall that the Fibonacci numbers $F(0), F(1) \ldots$ are given by $F(0) = F(1) = 1$ and the recurrence

$$F(n+1) = F(n) + F(n_1), \qquad n \geq 2.$$

Show that for any $n \geq 0$, $\gcd(F(n+1), F(n)) = 1$.

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12. **Representing polynomials**

Let $f$ be a polynomial of degree at most $d$. The *coefficient representation* of $f$ is the sequence $(a_0, a_1, \ldots, a_d)$ of coefficients of $f$. A *point-value representation* of $f$ is a collection $\{(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_t, f(x_t))\}$ of values of $f$ at any $t$ points $x_1, x_2, \ldots, x_t$, where $t \geq d+1$. (Recall that a polynomial of degree $d$ is completely determined by its values at any $d+1$ points. Note that $t$ may be greater than $d+1$, so more points than necessary may be given.)

In the following questions, let $f$ and $g$ be any two real polynomials of degree at most $d$.

(a) What is the maximum degree of the product polynomial $fg$?

(b) Given coefficient representations of $f$ and $g$, explain how to compute the coefficient representation of $fg$ using $O(d^2)$ arithmetic operations (additions/subtractions/multiplications/division over real numbers.

(c) Now suppose that $f$ and $g$ are specified by point-value representations at $t$ points for some $t \geq d+1$, i.e., $f$ is specified as $(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_t, f(x_t))$, and $g$ as $(x_1, g(x_1)), (x_2, g(x_2)), \ldots, (x_t, g(x_t))$. With a suitable value of $t$ (which you should specify), show how to compute a point-value representations of $fg$ using only $O(d)$ arithmetic operations.

(d) Suppose that polynomial $g$ divides polynomial $f$, and that $f, g$ are given in point-value representation as in part (c) with $t = d+1$. Show how to compute a point-value representation for the quotient $f/g$ using $O(d)$ arithmetic operations, and justify your algorithm carefully.

(e) Suppose you are given $f$ in coefficient representation, and you want to compute a point-value representation for $f$ at $t = d+1$ points. Show how to do this using $O(d^2)$ arithmetic operations. [Hint: Show how to evaluate $f$ at one point using $O(d)$ operations; to do this, consider writing $f$ in the form $f(x) = a_0 + xh(x)$, where $h$ is polynomial of degree at most $d-1$, and iterating.]