Implications

$P \Rightarrow Q$

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$	$Q \Rightarrow P$
0	0	1	1	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

Contrapositive: $\neg Q \Rightarrow \neg P$

$\neg(\neg Q) \vee (\neg P) \equiv \neg P \vee Q$

Converse: $Q \Rightarrow P$

$\forall$ vs. $\exists$

"for all" "there exists"

$\neg\forall \equiv \exists$ $\neg\exists \equiv \forall$

Proof is all-encompassing; disprove by counterexample

Proof by example; disprove by conventional proof

Optimality

Theorem: The pairing output by the male propose-and-reject algorithm is male optimal.

Suppose for contradiction that the pairing is not male optimal. Let day k be the first day when some man M was rejected by his optimal woman W in favor of M*. Because k is the first such day, we know M* has not been rejected by his optimal woman, who is at most as liked as W. But we know by the definition of optimal woman that there exists a stable pairing (M, W), which can't be possible if W prefers M* to M. Contradiction.

Theorem: If a pairing is male optimal, it is female pessimal.

Let T be a male optimal pairing where (M, W) exists. Now suppose for contradiction that M is not W's pessimal man, that there exists an inferior M' who can be stably paired with W in some other stable pairing. Since W prefers M to M', that pairing S cannot possibly be stable because W is M's optimal woman. Contradiction. So any male optimal pairing must also be female pessimal.

Types of Proofs

Theorem: True proposition guaranteed by proof

conjecture: Cannot prove, educated guess

lemma: Small theorem to use in proofs

axiom: Statement accepted as true w/o proof

Direct Proof

$P \Rightarrow Q$, assume P... therefore Q

Contraposition

Assume $\neg Q$ ... therefore $\neg P$,

so $\neg Q \Rightarrow \neg P \equiv P \Rightarrow Q$

Contradiction

Assume $\neg P$ ... R ... $\neg R$

Contradiction, therefore P

Cases

Construct cases to encompass all scenarios

Propose-and-Reject

Algorithm

Morning: Man goes to first woman who has not been crossed off his list and proposes.

Afternoon: Each woman says "maybe" to man she likes best and "no" to all the rest.

Evening: Each rejected suitor crosses off the woman who rejected him.

Improvement Lemma: If a man M proposes to W on the kth day, then on every subsequent day she has someone on a string she likes at least as much as M.

Proof: Suppose for contradiction that on day j > k that she has some M' on a string she likes less than M. On day j-1 she has some M* she likes at least as much as M, who will propose to her again. She must have rejected M* for M', which violates the algorithm. Contradiction.

Lemma: The algorithm terminates with a pairing.

Suppose for contradiction that there exists a man M who does not have a partner after the algorithm terminates. He must have proposed to all n women on his list, so all n women must have someone on a string (n men). So, there are n+1 men. Contradiction.

Theorem: The pairing produced by the algorithm is always stable.

Consider any couple (M, W) in the final pairing and suppose that M prefers some W' to W. Since W' occurs earlier in his preference list, he must have proposed to her first. Since they are not paired, she must have rejected him for some M* she likes better than M. By the Improvement Lemma, her final partner will be at least as liked as M*, so she will not prefer M. No man can be involved in any rogue couple.

Induction

$P(0) \wedge P(n) \Rightarrow P(n+1)$

Make sure that the inductive hypothesis is used to reach the inductive step / induct step can easily be modified to reach the n stage

Make sure to look for edge cases in your assumptions

Strong Induction $\rightarrow$ $P(0) \wedge P(1) \ldots \wedge P(n) \Rightarrow P(n+1)$

Essentially equivalent by proof of $\forall n P(n)$

Take $Q(n) = P(0) \wedge P(1) \ldots \wedge P(n)$

Now, prove $\forall Q(n)$ by simple induction, where

$Q(0) \wedge Q(n) \Rightarrow Q(n+1)$

which shows the same thing as a strong induction proof of $\forall n P(n)$

Strengthening the Inductive Hypothesis

Well-ordering Principle

If $S \subseteq \mathbb{N}$ and $S \neq \emptyset$ then S has a smallest element.

Equivalence of WOP and induction:

Induction using WOP:

WOP allows you to select a smallest element. Induction on the natural numbers, WOP allows you to establish a base case case that begins your induction

WOP using Induction:

Base case: Consider an S of size 0 or an S of size 1. We can select the smallest element from S.

Inductive Hypothesis: Suppose that for all $1 \leq k \leq n$ that we can select the smallest element from any set S of size k.

Induction step: Consider a set S of size $n+1$. We can remove one item E from the set to reduce it to size n. From this set S', we know that we can select the smallest element E'. Now, comparing E and E', if E is smaller, then E is the smallest in S. Otherwise, E' is the smallest in S.

For infinite S:

Pick an element γ of S and apply the filter $< \gamma$ to S, $S' = \{x \in S \mid x < \gamma\}$. By property of natural numbers and thus has a minimal element β, which is smaller than all elements in S^c and a member of S.