Mathematical Induction

Victor Adamchik

Fall of 2005

Lecture 1 (out of three)

■ Plan

1. The Principle of Mathematical Induction
2. Induction Examples

■ The Principle of Mathematical Induction

Suppose we have some statement $P(n)$ and we want to demonstrate that $P(n)$ is true for all $n \in \mathbb{N}$. Even if we can provide proofs for $P(0)$, $P(1)$, ..., $P(k)$, where k is some large number, we have accomplished very little. However, there is a general method, the Principle of Mathematical Induction.

Induction is a defining difference between discrete and continuous mathematics.

Principle of Induction. In order to show that $\forall\, n,\ P(n)$ holds, it suffices to establish the following two properties:

(I1) Base case: Show that $P(0)$ holds.

(I2) Induction step: Assume that $P(n)$ holds, and show that $P(n+1)$ also holds.

In the induction step, the assumption that $P(n)$ holds is called the Induction Hypothesis (IH). In more formal notation, this proof technique can be stated as

$$[P(0) \wedge \forall\, k\, (P(k) \Longrightarrow P(k+1))] \longrightarrow \forall\, n\, P(n)$$

You can think of the proof by (mathematical) induction as a kind of recursive proof:

Instead of attacking the problem directly, we only explain how to get a proof for $P(n+1)$ out of a proof for $P(n)$.

How would you prove that the proof by induction indeed works??

Proof (by contradiction) Assume that for some values of n, $P(n)$ is false. Let n_0 be the least such n that $P(n_0)$ is false. n_0 cannot be 0, because $P(0)$ is true. Thus, n_0 must be in the form $n_0 = 1 + n_1$. Since $n_1 < n_0$ then by $P(n_1)$ is true. Therefore, by inductive hypothesis $P(n_1 + 1)$ must be true. It follows then that $P(n_0)$ is true.
Contradiction.

The crucial part of this proof is that we are able to get a least element. This is called the Least Element Principle. We have been using this principal a few times when we cover the divisibility.

Comments

The base case can start with any nonnegative number. If that number is n_0 then you would prove that assertion $P(n)$ holds for all $n \geq n_0$.

$$[P(n_0) \wedge \forall\, k \geq n_0\, (P(k) \Longrightarrow P(k+1))] \longrightarrow \forall\, n\, P(n)$$

The induction step not necessarily should start with n. You can change the step from $n-1$ to n, where $n > 0$. Sometimes this yields slightly shorter expressions. However, you cannot make a step from $n-1$ to $n+1$.

Induction Examples

First Example

Prove for $n \geq 1$

$$1 * 1! + 2 * 2! + 3 * 3! + \ldots + n * n! = (n+1)! - 1$$

This could be also written by using Σ notation

$$\sum_{k=1}^{n} k\,k! = (n+1)! - 1$$

If you take 15-355 (http://www.andrew.cmu.edu/course/15-355/) you will learn a more general approach for deriving and proving combinatiral identities.

Proof.

Base case: $n = 1$

The left hand side is 1*1!. The right hand side is 2! - 1. They are equal.

Inductive hypothesis. Suppose this holds

$$\sum_{k=1}^{n} k\,k! = (n+1)! - 1$$

We need to prove

$$\sum_{k=1}^{n+1} k\,k! = (n+2)! - 1$$

Consider the left hand side

$$\sum_{k=1}^{n+1} k\,k! = \sum_{k=1}^{n} k\,k! + (n+1) * (n+1)! =$$

$$(n+1)! - 1 + (n+1) * (n+1)! =$$

$$(n+1)!\,(1 + n + 1) - 1 = (n+2)! - 1$$

QED

The ATM Machine

Suppose an ATM machine has only two dollar and five dollar bills. You can type in the amount you want, and it will figure out how to divide things up into the proper number of two's and five's.

Claim. The ATM machine can generate any output amount $n \geq 4$.

Proof:

By induction on n.

Base case: $n = 4$. 2 two's, done.

Induction step: suppose the machine can already handle $n \geq 4$ dollars. To produce $n + 1$ dollars, we proceed as follows.

Case 1: The n dollar output contains a five. Then we can replace the five by 3 two's to get $n + 1$ dollars.

Case 2: The n dollar output contains only two's.

Since $n \geq 4$, there must be at least 2 two's. Remove 2, and replace them by 1 five.

Tower of Hanoi

You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top on each other according to size on the first peg -- the largest disk being on the bottom, and the smallest on the top. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissable moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg.

Claim. It takes $2^n - 1$ moves to move n disks from the first peg to the third peg.

Proof:

Base case: $n = 1$. It takes exactly one move. Indeed, $2^1 - 1 = 1$.

Inductive hypothesis: suppose it takes $2^n - 1$ moves to move n disks.

We need to prove that the number of moves for $n + 1$ disks is

$$2^{n+1} - 1$$

We proceed as follows. First, we move top n disks to the second peg (using the third peg as an intermediate peg.) It takes $2^n - 1$ moves. Then, we move the last disk to the third peg - it takes one move. Finally, we move n disks from the second peg to the third peg. Here is the total number of moves

$$(2^n - 1) + 1 + (2^n - 1) = 2 * 2^n - 1 = 2^{n+1} - 1$$

All horses are the same color

Claim. All horses are the same color.

Proof.
The proof by induction on the number of horses.

Base: if there is one horse, then it is trivially the same color as itself.

Inductive hypothesis: assume that horses, numbered from 1 to n have the same color.

Inductive step: There are $n + 1$ horses. By the induction hypothesis, horses from 1 to n have the same color, let's say white. In particular, horse 2 is white. Also by induction hypothesis 2 through $n + 1$ have the same color - white, since 2 is white. Therefore, all of the horses have the same color.

Explanation.
We deal with two sets of horses
- from 1 to n. Denote this by G_1.
- from 2 to $n + 1$ Denote this by G_2.

Both of them have n horses and thus we conclude that horses in the union $G_1 \bigcup G_2$ have the same color.

It would be true if their intersection $G_1 \bigcap G_2$ is NOT empty. And this is a flaw! If G_1 and G_2 are disjoint, then there is no reason why horses from both sets have the same color. When these two sets are disjoint? When $n = 1$. Each set has only one element. $P(1)$ does not imply $P(2)$.

Exercise (stamp theorem). Prove by induction that for any integer $n > 23$ there exist non-negative integers x and y such that

$$n = 7\,x + 5\,y$$

Exercise (nested radicals).

(a) Prove by induction that for any integer $n \geq 2$ the following number

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$$

is irrational. The above radical signs contain exactly n ones.

(b) Can you discover that irrational number?