Properties of CTFT

Analysis Equation

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt$$

Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \, d\omega$$

Time Shift

$$x(t-t_0) \xleftrightarrow{Ft} e^{-j\omega t_0} X(j\omega)$$

Conjugation

$$x^*(t) \xleftrightarrow{Ft} X^*(-j\omega)$$

if $x(t) \in \mathbb{R}$,

$$X(j\omega) = X^*(-j\omega)$$

Differentiation & Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{Ft} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau) \, d\tau \xleftrightarrow{Ft} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$$

Time & Freq. Scaling

$$x(at) \xleftrightarrow{Ft} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

if x is even, $X(j\omega) \in \mathbb{R}$,

if x is odd, $X(j\omega) \in \mathbb{I}$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Differentiation in Freq.

$$t x(t) \xleftrightarrow{Ft} j \frac{d}{d\omega} X(j\omega)$$

Properties of DTFT

Analysis Equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Synthesis Equation

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

Differencing (DT v. of Diferentiation)

$$x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\omega}) X(e^{j\omega})$$

Accumulation (DT v. of integration)

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta[\omega - 2\pi k]$$

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(e^{j\omega})|^2 d\omega$$

Properties of CTFS

Synthesis Equation: $\omega_0 = \frac{2\pi}{N}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Analysis Equation

$$a_k = \frac{1}{N} \int_{\langle N \rangle} x(t) e^{-jk\omega_0 t} \, dt$$

$x^*(t) \Rightarrow a^*_{-k}$

$x(-t) \Rightarrow a_{-k}$

$x(t) \in \mathbb{R} \Rightarrow a_k = a^*_{-k}$

$x(t) \in \mathbb{R}$ & even

a_k real & even

$x(t) \in \mathbb{R}$ & odd

a_k imagin. & odd

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Properties of DTFS

Synthesis Equation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \qquad \omega_0 = \frac{2\pi}{N}$$

Analysis Equation

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

Conjugate, real, even & odd same as CT

Parseval's

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

a_k periodic w/ period N

Half-Wave Symmetry

$$x(t) = -x\left(t + \frac{T}{2}\right) \quad \forall t$$

then $a_k = 0$ if k is even

GLP:

$$H(e^{j\omega}) = \underbrace{A(e^{j\omega})}_{\text{Real, can be neg.}} e^{-j\alpha\omega + \beta}$$

LP: $A(e^{j\omega})$ is Real, nonneg., $\beta = 0$

2D CTFT Analysis

$$X(j\omega_1, j\omega_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t_1, t_2) e^{-j\omega_1 t_1} e^{-j\omega_2 t_2} \, dt_1 \, dt_2$$

2D CTFT Synthesis

$$x(t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(j\omega_1, j\omega_2) e^{j\omega_1 t_1} e^{j\omega_2 t_2} \, d\omega_1 \, d\omega_2$$

2D DTFT Analysis

$$X(j\omega_1, j\omega_2) = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} x[n_1, n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

2D DTFT Synthesis

$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(j\omega_1, j\omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} \, d\omega_1 \, d\omega_2$$

Separability:

$$x[n_1, n_2] = x_1[n_1] \, x_2[n_2]$$

then $\updownarrow \mathcal{F}$

$$X(j\omega_1, j\omega_2) = X_1(j\omega_1) X_2(j\omega_2)$$

Projection Slice Theorem

$$x_0(t_1) \triangleq \int_{-\infty}^{+\infty} x(t_1, t_2) \, dt_2$$

$$X_0(j\omega_1) = X(j\omega_1, j\omega_2) \Big|_{\omega_2 = 0}$$

Causality:

$$h[n] = 0 \quad \forall n < 0$$

Stability:

$$\sum_{n=-\infty}^{+\infty} x[n] < \infty$$

$$\int_{-\infty}^{+\infty} x(t) \, dt < \infty$$

LCCDE:

$$y[n] + a_1 y[n-1] + \ldots + a_k y[n-k] = b_0 x[n] + \ldots + b_l x[n-l]$$

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + \ldots b_l e^{-jl\omega}}{1 + a_1 e^{-j\omega} + \ldots + a_k e^{-jk\omega}}$$

$$h[n] = h_e[n] + h_o[n]$$

where

$$h_e[n] = (h[n] + h[-n]) \tfrac{1}{2}$$

$$h_o[n] = (h[n] - h[-n]) \tfrac{1}{2}$$

true for DT

CT - Fourier Pairs

Signal	Fourier Trans.	Fourier Coeff.
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$	$a_1 = 1$, $a_k = 0$
$x(t) = 1$	$2\pi \delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T} \ \forall k$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin(\omega T_1)}{\omega}$	—
$\frac{\sin(Wt)}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at} u(t)$, $Re\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$t e^{-at} u(t)$, $Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, $Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

DT - Fourier Pairs

Signal	Fourier Trans.	Fourier Coeff.
$\sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 t}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	$\omega_0 = \frac{2\pi m}{N}$; $a_k = \begin{cases} 1 & \text{if } k = m, m \pm N \ldots \\ 0 & \text{o.w.} \end{cases}$; if $\omega_0 \neq \frac{2\pi m}{N}$, Not periodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N \ldots \\ 0 & \text{o.w.} \end{cases}$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N} \ \forall k$
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert < N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin(Wn)}{\pi n}$, $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ periodic w/ period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1) a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—