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Signal – function (of time)

System $\cdot\ x(t) \to \boxed{H} \to y(t)$ function on a signal

System types:

Info retrieval (radio)
- find x
- given H
- measure y

Control (airplane)
- design x
- given H
- desired y

System ID (MRI)
- given x
- find H
- measure y

Signal Processing (Filters)
- given x
- design H
- desired y

LTI systems:

Linear $\quad ax(t) \to ay(t)$

$\quad x_1(t)+x_2(t) \to y_1(t)+y_2(t)$

TI (Time Invariance) $\quad x(t-T) \to y(t-T)$

$d[n]$ – unit impulse (Discrete) (Kronecker Delta)

IF LTI, $h[n] = H\{d[n]\}$ represents system

$x[n] = \dots + x[-1]d[n+1] + x[0]d[n] + x[1]d[n-1] + \dots$

$= \sum_k x[k]d[n-k]$

by LTI,

$y[n] = H\{x[n]\} = \sum_k x[k]h[n-k]$ (convolution sum)

$d(t)$ – unit impulse (Continuous) (Dirac Delta)

$\int d(t) = 1$

$d(k) = 0 \ \forall k \neq 0$

IF LTI, $h(t) = H\{d(t)\}$ represents system

$y(t) = H\{x(t)\} = \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$ (convolution integral)

$:= x(t) * h(t)$

Convolution – properties:

- commutative $\quad x*h = h*x \quad$ (change of variables)
- distributive $\quad x*(h_1+h_2) = x*h_1 + x*h_2 \quad$ (parallel x → h_1, h_2 → ⊕ →)
- associative $\quad x*(h_1*h_2) = (x*h_1)*h_2 \quad$ (series $x \to h_1 \to h_2$)

(thus, these are properties of LTI systems)

Causal Systems → $y(n)$ only depends on $x(k)$, $k \leq n$ (current/previous inputs) * realtime systems are causal

→ (if LTI) $h(t) = 0 \ \forall t < 0$ (if not, $y[n] = \sum_k x[k]h[n-k]$ would turn on for $x[k], n<k$)

BIBO stable systems → bdd input (i.e. $u(t)$) ⇒ bdd output (*recall: bdd → $|x(t)| < M \ \forall t$) (*good systems are BIBO stable, if finite input, get finite output)

→ (if LTI) $\int_{-\infty}^{\infty} |h(T)|\,dT < \infty$

Sufficiency: $y(t) = \int h(T)x(t-T)\,dT < \int h(T) B\,dT \leq B\int |h(t)|\,dT < \infty$ ✓

Necessity: if $\int_{-\infty}^{\infty} |h(T)| = \infty$, let $x[n] = $ sign of $h[-n]$, then $y[n] = \sum h[k]x[n-k]$

$y[0] = \sum h[k]x[-k] = \sum |h[k]| = \infty$ unbounded ✓

System → LTI? (if so, can use $h(t)$)

↙ Causal? ↓ Stable?

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$(L \vee TI) \wedge C \rightarrow \neg P$

(linear or TI) and causal ⇒ output doesn't precede input

$P \rightarrow \neg C \vee \neg(L \vee TI)$

output precedes input ⇒ not causal or not linear and not TI

LCC Differential Equation

$$x(t) \rightarrow \boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}} \rightarrow y(t)$$

- causal ✓
- LTI iff $y(0) = y'(0) = \dots = y^{N-1}(0) = 0$ (zero initial conditions)

LCC Difference Equation

$$x[n] \rightarrow \boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]} \rightarrow y[n]$$

$$y(t) = \underbrace{y_{ZSR}(t)}_{\text{forced response}} + \underbrace{y_{ZIR}(t)}_{\text{initial conditions}}$$

- (causal iff $a_0 \neq 0$)
  assume $a_0 = 1$ (wlog; just divide by a_0)
- (assuming causal) LTI iff $y[-1] = y[-2] = \dots = y[-N] = 0$ (zero initial conditions)

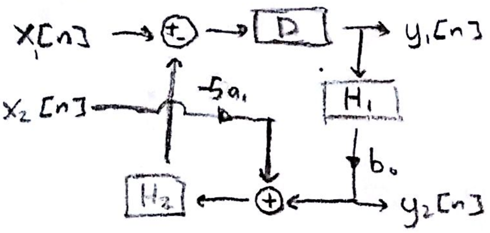
Finite Impulse Response (FIR) input $\delta[n]$ → output has finite duration (LTI... $h[n]$ has finite duration)

defined by LCC Difference Eq., $N=0$: $y[n] = b_0 x[n] + \dots + b_M x[n-M]$ (LTI)

(no feedback needed!) (stable!)

Infinite Impulse Response (IIR) input $\delta[n]$ → output has infinite duration (LTI... $h[n]$ has infinite duration)

Block Diagrams



Signals $x[n], \dots$

Systems $\boxed{H}$, $\boxed{D}$ (delay), ...

mult. $\xrightarrow{c_0}$

addition $\oplus$, $\ominus$

LTI systems + complex exponentials

- eigenvectors? $x(t) = e^{st}$ $s \in \mathbb{C}$, $s = \sigma + j\omega$ $\quad x(t) = \underbrace{e^{\sigma t}}_{\text{envelope}} \underbrace{e^{j\omega t}}_{\text{periodic}}$

  $x[n] = z^n$ $z \in \mathbb{C}$, $z = re^{j\omega}$ $\quad x[n] = r^n e^{j\omega n}$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = \underbrace{\left(\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau\right)}_{H(s)} e^{st}$$

$$y(t) = H(s)\, x(t)$$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$$= \sum_k h[k] z^{n-k} = \underbrace{\left(\sum h[k] z^{-k}\right)}_{H(z)} z^n$$

$$y[n] = H(z)\, x[n]$$

Frequency Response of LTI systems

$s = j\omega$ $\quad y(t) = H(j\omega) e^{j\omega t}$ (input is complex exponential)

$z = e^{j\omega}$ $\quad y[n] = H(e^{j\omega}) e^{j\omega n}$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$H(e^{j\omega}) = \sum_k h[k] e^{-j\omega k}$$

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Fourier Series

CTFS (Continuous time)

(periodic signals: $x(t+T)=x(t)\ \forall t$)

Synthesis $\boxed{x(t)=\sum_k a_k e^{jk\omega_0 t}}$ $\omega_0=\frac{2\pi}{T}$

$k=0$: DC value
$k=\pm1$: 1^{st} harmonic

$x^*(t)=\sum_k a_k^* e^{-jk\omega_0 t}=\sum_k a_{-k}^* e^{jk\omega_0 t}$

real signals: $x(t)=x^*(t)$
$a_k=a^*_{-k}$

Analysis $\boxed{a_n=\frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt}$

$Ta_n=\sum_k a_k\left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]$ $k=n \rightarrow T=\int_0^T 1\,dt$; $k\neq n \rightarrow 0=\int_0^T \sin(t)$ (over periods)

DC value: $a_0=\frac{1}{T}\int_0^T x(t)\,dt$

Parseval's Thm

average power $=\sum_k |a_k|^2$

$\sum_n |x[n]|^2=\sum_n \left|\sum_k a_k e^{jk\omega_0 t}\right|^2=\sum_k a_k a_{-k}+\cancel{\text{crossterms}}^{\,0}=\sum_k |a_k|^2$

total harmonic distortion

$THD=\sqrt{\frac{\sum_{k=2}^{\infty}|a_k|^2}{|a_1|^2}}$

DTFS Fourier Series (Discrete time)

(periodic signals: $x[n+N]=x[n]\ \forall n$)

Synthesis $\boxed{x[n]=\sum_0^{N-1} a_k e^{jk\omega_0 n}}$ $\omega_0=\frac{2\pi}{N}$

$e^{j(k+N)\omega_0 n}=e^{jk\omega_0 n}$

Properties:
- $e^{jk\omega_0(n+N)}=e^{jk\omega_0 n}$
- $e^{j(k+N)\omega_0 n}=e^{jk\omega_0 n}$
- $\sum_0^{N-1} e^{jk\omega_0 n}=\begin{cases}N & \text{if } k=0,\pm N,\pm 2N,\dots\\ 0 & \text{otherwise}\end{cases}$

real signals: $x[n]=x^*[n]$
$a_k=a^*_{-k}=a^*_{N-k}$

Analysis $\boxed{a_k=\frac{1}{N}\sum_0^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}}=\frac{1}{N}\sum_0^{N-1}x[n]e^{-jk\omega_0 n}$

FT	DTFT
cts t ↔ cts w	discrete t ↔ periodic w
CTFS	**DTFS / DFT**
periodic t ↔ discrete w	periodic, discrete t ↔ periodic, discrete

FT Fourier Transform

"aperiodic = periodic with $T=\infty$"

- take CTFS, $T\rightarrow\infty$ $\omega_0=\frac{2\pi}{T}\rightarrow 0$
- $Ta_k=\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt$
  $=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\Big|_{\omega=k\omega_0}$
  $:=X(j\omega)$

Analysis $\boxed{X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt}$

Synthesis $\boxed{x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega}$

since $\tilde{x}(t)=\frac{1}{2\pi}\sum_k \underbrace{\frac{2\pi}{T}}_{\omega_0} X(j\omega)e^{j\omega t}\Big|_{\omega=k\omega_0}$ (Fourier series) $=\sum_k a_k e^{jk\omega_0 t}=\sum_k \frac{1}{T}X(j\omega)e^{jk\omega_0 t}$

Convergence (Dirichlet conditions) of $\frac{1}{2\pi}\int_{-W}^{W}X(j\omega)e^{j\omega t}d\omega$ $=x(t)$ as $W\rightarrow\infty$
$=$ average (at discontinuities)

1. $\int_{-\infty}^{\infty}|x(t)|dt<\infty$ (stability)
2. Finite interval ⇒ finite # of minima, maxima
3. finite interval ⇒ finite # of discontinuities, all finite

- $\measuredangle H(j\omega)=\tan^{-1}\left(\frac{Im}{Re}\right)$ = phase
- $|H(j\omega)|$ = magnitude

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Fourier Transform examples

- $e^{-at}u(t)\quad a>0 \longleftrightarrow \frac{1}{a+jw}$
- $\delta(t) \longleftrightarrow 1$
- $\text{rect}(t/2T_1) \longleftrightarrow \frac{2\sin(wT_1)}{w}$
- $\frac{\sin(Wt)}{\pi t} \longleftrightarrow \text{rect}(w/2W)$ (LPF)
- $\sum_k a_k e^{jkw_0 t} \longleftrightarrow \sum_k 2\pi a_k \delta(w-kw_0)$
- $\cos(w_0 t) \longleftrightarrow \pi(\delta(w-w_0)+\delta(w+w_0))$
- $\sum_k \delta(t-kT) \longleftrightarrow \frac{2\pi}{T}\sum_k \delta(w-kw_0),\ w_0=\frac{2\pi}{T}$ (impulse train)
- $\frac{1}{2\pi}e^{jw_0 t} \longleftrightarrow \delta(w-w_0)$
- discrete $\longleftrightarrow$ periodic
- periodic $\longleftrightarrow$ discrete

FT properties

- $ax(t)+by(t) \longleftrightarrow aX(jw)+bY(jw)$
- $x(t-t_0) \longleftrightarrow e^{-jwt_0}X(jw)$
- $e^{jw_0 t}x(t) \longleftrightarrow X(j(w-w_0))$
- $x^*(t) \longleftrightarrow X^*(-jw)$
- $x'(t) \longleftrightarrow jwX(jw) - x(0)$
- $\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{X(jw)}{jw} + \pi X(0)\delta(w)$
- $x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{jw}{a}\right)$
- $x(-t) \longleftrightarrow X(-jw)$
- $\int |x(t)|^2 dt \longleftrightarrow \frac{1}{2\pi}\int |X(jw)|^2 dw$
- $x(0) = \frac{1}{2\pi}\int X(jw)dw$
- $X(0) = \int x(t)dt$
- $h(t)*x(t) \longleftrightarrow H(jw)X(jw)$
- $\frac{1}{2\pi}h(t)x(t) \longleftrightarrow H(jw)*X(jw)$

Frequency Response of LTI System

- $H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$
- (For convergence, must be BIBO stable!)

Freq. Response, LCC Differential Eq

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Rightarrow \sum_{k=0}^{N} a_k (jw)^k Y(jw) = \sum_{k=0}^{M} b_k (jw)^k X(jw)$$

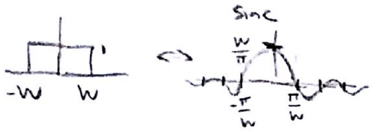
i.e. $\tau y'(t) + y(t) = x(t)$

$H(jw) = \frac{1}{1+\tau jw}$

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k}$$

Ideal LPF

(rect)



Problems.
- Infinite extent (IIR) $\rightarrow$ window in time! ($\cdot\,\text{rect}(t/2T_1)$)
- Non-causal $\rightarrow$ time shift! (linear phase $-wT_1$ from e^{-jwT_1})

Discrete Time Fourier Transform

DTFT

"aperiodic = periodic with $N=\infty$"

- take DTFS, $N\to\infty$, $w_0 = \frac{2\pi}{N} \to 0$
- $Na_k = \sum_{\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n} = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}\Big|_{w=kw_0}$
  $:= X(e^{jw})$

$$\tilde{x}[n] = \sum_{\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} = \frac{1}{2\pi}\sum_{\langle N\rangle}\frac{2\pi}{N}X(e^{jkw_0})e^{jkw_0 n} \to \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

w normalized $0 \le w \le 2\pi$, $w = wT$

Since $X(e^{jw})$ is periodic, only integrate over period

Analysis $\boxed{X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}}$

Synthesis $\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw}$

Convergence if $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ (like Dirichlet 1) stability!

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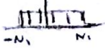
DTFT examples

- $a^n u[n] \quad |a|<1 \leftrightarrow \frac{1}{1-ae^{-j\omega}}$
- $\delta[n] \leftrightarrow 1$
-  $-N_1 \quad N_1$ $\leftrightarrow \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$
- $\frac{1}{2\pi} \leftrightarrow \sum_{\ell} \delta(\omega-2\pi\ell)$
- $e^{j\omega_0 n} \leftrightarrow \sum_{\ell} \delta(\omega-\omega_0-2\pi\ell)$
- $\sum_{\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow \sum_k 2\pi a_k \delta(\omega-\frac{2\pi}{N}k)$

DTFT properties

- $x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$
- $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
- $x[-n] \leftrightarrow X(e^{-j\omega})$
- $x^*[n] \leftrightarrow X^*(e^{-j\omega})$
- $x[n]-x[n-1] \leftrightarrow (1-e^{-j\omega})X(e^{j\omega}) \quad (=x'[n])$
- $\sum_m x[m] \leftrightarrow \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{\ell}\delta(\omega-2\pi\ell)$
- $n\,x[n] \leftrightarrow j\,X'(e^{j\omega})$
- $\sum |x[n]|^2 \leftrightarrow \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$
- $x_{(m)}[n] \leftrightarrow X(e^{j\omega m})$ (downsample... stretch) $0 \quad m \quad 2m \quad 3m \ldots$
- $h[n] * x[n] \leftrightarrow H(e^{j\omega})X(e^{j\omega})$
- $x_1[n]x_2[n] \leftrightarrow \frac{1}{2\pi} X_1(e^{j\omega}) * X_2(e^{j\omega})$

Freq. Response, LCC Difference Eq

- $\sum^N a_k y[n-k] = \sum^M b_k x[n-k]$ (plug in $x[n]=\delta[n]$, $y[n]=h[n]$)

$$H(e^{j\omega}) = \frac{\sum^M b_k e^{-j\omega k}}{\sum^N a_k e^{-j\omega k}}$$

i.e. $y[n]-\alpha y[n-1]=x[n]$.

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

Ideal LPF

Problems

- infinite extent → window in time (Gibbs phenomenon, tapered windows) Blackman, Bartlett, Hamming, Hanning

tradeoff: main lobe width vs side lobe amplitude

- causality → time delay!

Linear Phase

- $H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}$ (real, nonnegative) $\quad \angle H(e^{j\omega}) = -\alpha\omega$
- desired to prevent phase distortion!! (each frequency delayed by same duration)

Generalized linear Phase

- easier to implement in FIR design
- $H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta}$ (real, could be negative) $\quad$ phase $= \beta-\alpha\omega$, discontinuous jump by π when $A(e^{j\omega})$ changes signs
- windowed FIR systems!

time ↔ frequency $\quad \angle H(e^{j\omega})$

even ↔ real $\quad \alpha=0, \beta=0$

even, timeshifted N_1 ↔ $\alpha=N_1$, $\beta=0$

odd ↔ $\alpha=0$, $\beta=\pi/2$ (pull out $e^{j\pi/2}$)

odd, timeshifted N_1 ↔ $\alpha=N_1$, $\beta=\pi/2$

2DFT

Analysis
$$X(j\omega_1,j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}\,dt_1\,dt_2$$

Synthesis
$$x(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1,j\omega_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}\,d\omega_1\,d\omega_2$$

Convergence: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t_1,t_2)|\,dt_1\,dt_2 < \infty$ (stability)

Separability if $x[n_1,n_2] = x_1[n_1]x_2[n_2]$, then $X(e^{j\omega_1},e^{j\omega_2}) = X_1(e^{j\omega_1})X_2(e^{j\omega_2})$

Linear Shift-Invariant $\quad y[n_1,n_2] = x[n_1,n_2] * h[n_1,n_2]$; $\quad Y(e^{j\omega_1},e^{j\omega_2}) = X(e^{j\omega_1},e^{j\omega_2})H(e^{j\omega_1},e^{j\omega_2})$

Projection-Slice $\quad$ i.e. $x_0(t_1) = \int_{-\infty}^{\infty} x(t_1,t_2)\,dt_2$

$X_0(j\omega_1) = X(j\omega_1,j\omega_2)|_{\omega_2=0}$

- can project along any direction, 1D "slice" of 2DFT
- uses in medical imaging, slice on different angles to create 2D image

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Discrete Fourier Transform (DFT)

- discrete, periodic in time $\Leftrightarrow$ discrete, periodic in frequency
- take DTFS, look at one period in time $\Leftrightarrow$ one period in frequency

Analysis: $X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} = N a_k \quad (0 \le k \le N-1)$ (period $= N$)

recall: $a_k = a_{k+N}$ for Fourier series

Synthesis: $x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$ (periodic in time, discrete in frequency!)

$W_N = e^{-j\frac{2\pi}{N}}$

$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$

$\tilde{x}[n] = x[n \bmod N]$

$a_k = \frac{1}{N} X[k \bmod N]$

(DTFS) (DFT)

Properties:

- if $x[n] = x^*[n]$, $X[k] = X^*[n-k]$
- $DFT(\frac{1}{N} X[k]) = x[-n]$
- $x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]$
- $X[0] = \sum_{n=0}^{N-1} x[n]$

$X[k] = \frac{N}{2\pi} \text{ area} \{ X(e^{j\omega})|_{\omega = \frac{2\pi}{N}k} \}$

DT, D.Freq. (periodic)

DTFT vs DFT

DTFT = DTFS, $N \to \infty$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$

$\boxed{\star \text{ if duration of } x[n] \le N}$

$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \quad (0 \le k \le N-1) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$ (DTFT sampled N times, $k = 0, 1, \ldots, N-1$; frequency "bins" size $= \frac{2\pi}{N}$)

★ What if N < duration of $x[n]$?

$\hat{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \left(X(e^{j\omega})|_{\omega = \frac{2\pi}{N}k} \right) e^{j\frac{2\pi}{N}kn} \quad (0 \le n \le N-1)$

$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km} e^{j\frac{2\pi}{N}kn} = \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{k=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi}{N}k(n-m)} \right) = \ldots + x[n-N] + x[n] + x[n+N] + \ldots$

if $m = n \bmod N \Rightarrow 1$, otherwise $\Rightarrow 0$

$= \sum_{r=-\infty}^{\infty} x[n - rN]$

if N < duration of $x[n]$, signal aliased $\to$ not properly reconstructed in time

$\hat{x}[n] = x[n] + x[n+N] + \ldots \ne x[n]$

Convolution with DFT (Fast!!)

- LTI system $\Rightarrow \exists\, h[n]$ to represent system
- FIR system $\Rightarrow$ stable! causal!

$y[n] = \underbrace{x[n]}_{\text{duration } L} * \underbrace{h[n]}_{\text{duration } P} \qquad Y[k] = X[k] H[k]$

- Set $N \ge L + P - 1$ (duration of $y[n]$)
  - pad $x[n], h[n]$ with zeroes to make duration = N

- $y[n] = DFT^{-1}(DFT(x[n]) \cdot DFT(h[n]))$
  $= DFT^{-1}(X[k] H[k])$

★ convolution is $O(N^2)$
but DFT, DFT^{-1} is $O(N \log N)$ (i.e. FFT)

2D DFT

Analysis: $X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2}$

Synthesis: $x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}k_1 n_1} e^{j\frac{2\pi}{N_2}k_2 n_2}$

★ application: image compression (JPEG, etc)

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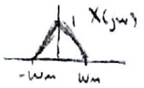
Sampling in time: $x_d[n] = x(nT)$ — can we recover $x(t)$ from samples? {using $X_d(e^{j\Omega})$? $\Omega = \omega T$ radians (normalized $0 \le \Omega \le 2\pi$)} DTFT

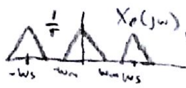
Shannon-Nyquist: yes!
- bandlimit $x(t)$ ($X(j\omega) = 0$ for $|\omega| > \omega_m$)
- Sample at $\omega_s > 2\omega_m$ ($\omega_s = \frac{2\pi}{T}$) then $x(t)$ uniquely determined by samples $x(nT)$!

★ extends to multiple dimensions, 2D sampling, etc.

How?
- let $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ impulse train, $x_p(t) = x(t)\cdot p(t)$
- $P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$ $\quad X_p(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

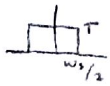
$\boxed{\omega_s = \frac{2\pi}{T}}$ $\quad \boxed{X_d(e^{j\Omega}) = X_p(j\omega)\big|_{\Omega = \omega T}}$



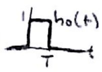


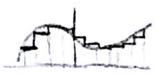
- Impulse train = repeater
  - if $\omega_s > 2\omega_m$, no overlap!!
- aliasing in frequency ∴ frequency overlap!

phase shift ⇒ reversal
★ wagon wheel effect
seems to spin slowly backwards
$\omega_{wo} > \frac{1}{2}$ eye sampling (~20Hz)

Exact Reconstruction
- $H(j\omega) =$ (LPF)  time domain: $h(t) = T\frac{\sin(\frac{\omega_s}{2}t)}{\pi t}$ $(T = \frac{2\pi}{\omega_s})$ 
- Sinc's are hard... (infinite extent, causality) but this works exactly $x_r(t) = h(t) * x_p(t) = \sum_n x(nT)h(t-nT)$

interpolation!

Approximate Reconstruction
- try  window: (rect in time) sinc in frequency! (we want rect)  spectral leakage

(0th hold)  Zero-order reconstruction

- try  (tapered window) sinc^2 in frequency! (we want rect) 

(1st hold) (linear)  First-order reconstruction

★: other filters to approximate sinc in time (LPF) – Hamming, etc. Application: Doppler ultrasound/radar

DT processing, CT signals
- $x(t) \rightarrow \boxed{C/D} \xrightarrow{x_d[n]} \boxed{\text{DT System}} \xrightarrow{y_d[n]} \boxed{D/C} \rightarrow y(t)$

  C/D: (ω_s) sample, $\omega \to \frac{\Omega}{T}$; DT System: $H_d(e^{j\Omega})$; D/C: LPF, $\Omega \to \omega T$, reconstruction (ω_s)

$\boxed{\text{A NOT LTI!!}}$

- if $x(t)$ bandlimited at $\frac{\omega_s}{2}$, $H_{eff}(j\omega) = \begin{cases} H_d(e^{j\omega T}) & |\omega| < \omega_s/2 \\ 0 & \text{otherwise}\end{cases}$

$\boxed{\Omega \text{ normalized } [0, 2\pi) \quad \Omega = \omega T}$

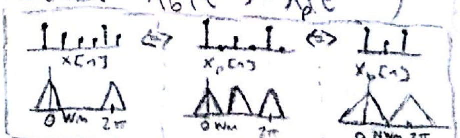
★ example: digital differentiator, non-integer delay (interpolation)
★ FIR approximations by windowing in time

Sampling DT signals
- $x_p[n] = x[n]p[n]$ $\quad p[n]$ is impulse train $= \sum_{k=-\infty}^{\infty}\delta[n-kN]$

  $= x[kN]\delta[n-kN]$ $\quad P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$ $\quad \boxed{\omega_s = \frac{2\pi}{N}}$
- $X_p(e^{j\omega}) = \frac{1}{2\pi}X(e^{j\omega}) * P(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}X(e^{j(\omega - k\omega_s)})$ (periodic in frequency)

Reconstruction identical to CT with sampled $h[n]$ (impulse response)

$\boxed{\downarrow N}$ Downsampling
- $x_b[n] = x[Nn]$ take each N^{th} sample, discard rest $\quad X_b(e^{j\omega}) = X_p(e^{j\omega/N})$
- Sampling CT at lower rate!



aliasing if $\omega_m N > \pi$ — bandlimit (prevent by killing high frequency before downsampling)

$\boxed{\uparrow N}$ Upsampling
- Expand signal, then interpolate → reconstruction LPF (i.e. frequency overlap)

  $x_p[n] = \begin{cases} x_b[n/N] & n = 0, \pm N, \ldots \\ 0 & \text{otherwise}\end{cases}$
- Sampling CT at faster rate (if no aliasing)!
- Only works if $x_b[n]$ could reconstruct $x(t)$...

$x[n] \rightarrow \boxed{\downarrow N} \rightarrow x_b[n]$

$x[n] \leftarrow \boxed{\uparrow N} \leftarrow x_b[n]$

UPSAMPLE first!! to preserve data: ↑, then ↓ (don't throw away samples)

$\rightarrow\boxed{\uparrow P/Q}\rightarrow \; = \; \rightarrow\boxed{\uparrow P}\rightarrow\boxed{\downarrow Q}\rightarrow$