HOW TO OPTIMIZE THAT SHIT!

Linear programs:

General form: $\min_x c^T x + d : Gx \le h,\ Ax = b$

Standard form: $\min_x c^T x : Ax = b,\ x \ge 0$

Tricks for standardization:

① $\max_x c^T x \Leftrightarrow \min_x (-c)^T x$

② $Ax \le b \Leftrightarrow Ax + s = b,\ s \ge 0$

③ If x is unrestricted in sign,
- introduce $x^+, x^- \ge 0$
- replace x by $(x^+ - x^-)$

Least squares:

$\min_x \|Ax - y\|_2^2 \quad x^* = (A^T A)^{-1} A^T y$

Variants:

$\min_x \|Ax - y\|_2^2 : Cx = d$

$\to \min_z \|\hat{A} z - \tilde{y}\|_2^2$

where $\hat{A} = AN$, $\tilde{y} = y - Ax_0$

N spans $\mathcal{N}(C)$, x_0 solves $Cx = d$

$\min_x \|Ax - y\|_2^2 + \lambda \|x\|_2^2$

$\to \min_x \|\hat{A}x - \tilde{y}\|_2^2, \quad \hat{A} = \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix}, \ \tilde{y} = \begin{bmatrix} y \\ 0 \end{bmatrix}$

$\min_x \|Ax - y\|_2^2 + x^T W x, \quad W \succ 0$

$x^* = (A^T A + W)^{-1} A^T y$

$\min_x \|x\|_2 : Ax = y$

if A is full row rank, $x^* = A^T (AA^T)^{-1} y$

SOCP:

Standard form:

$\min_x c^T x : \|A_i x + b_i\|_2 \le c_i^T x + d_i$

rotated SOC:

$\|x\|_2^2 \le yz \Longleftrightarrow \left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\|_2 \le y + z$

for $y, z \ge 0$

$x^T Q x + c^T x \le t \Leftrightarrow w^T w \le y,\ w = Q^{1/2} x,\ y = t - c^T x$

$Q \succeq 0$

Find projection z of x onto $\{x_0 + tu : t \in \mathbb{R}\}$

$\min_t \|x - x_0 - tu\|_2^2 \to t^* = u^T (x - x_0)$

$z = x_0 + t^* u$

For hyperplane $H = \{x : a^T x = b\}$

if $\|a\|_2 = 1$, projection of 0 on H is ba,

solving $\min_x \|x\|_2 : x \in H$

For affine set $\{x : Ax = b\}$

$x^* = A^T (AA^T)^{-1} b$

Quadratic Programs:

Standard form: $(H \succeq 0)$

$\min_x \frac{1}{2} x^T H x + c^T x : Gx = h,\ Ax \le b$

Constrained LS: $\min_x \|Rx - y\|_2^2 : Gx = h,\ Ax \le b$

transforms to standard with $H = 2R^T R \succeq 0$, $c^T = -2y^T R$

$\underline{\text{SVD}}$: $A = U\tilde{\Sigma}V^T$, $A \in \mathbb{R}^{m\times n}$

$$\overset{\longleftarrow m \longrightarrow}{m\updownarrow\begin{bmatrix} \underbrace{u_1 \cdots}_{\substack{r: \\ \text{range}(A)}} & \underbrace{\cdots u_m}_{\substack{m-r: \\ \text{null}(A^T)}} \end{bmatrix}} \; m\updownarrow \overset{\longleftarrow n \longrightarrow}{\begin{bmatrix} \underset{\leftarrow r \rightarrow}{\Sigma} \updownarrow r & 0 \\ 0 & 0 \end{bmatrix}} \; n\updownarrow \overset{\longleftarrow n \longrightarrow}{\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}} \begin{array}{l} \}r: \text{range}(A^T) \\ \}n-r: \text{null}(A) \end{array}$$

$\Sigma = \text{diag}(\sigma_1 \ldots \sigma_r)$

$\sigma_i^2 = \lambda_i(A^TA) = \lambda_i(AA^T)$

v_i: eigenvectors of A^TA, $Av_i = \sigma_i u_i$

$\|A\|_F^2 = \text{Tr}(A^TA) = \sum_i \sigma_i^2$

$\mathcal{E}(A) = \{Ax : \|x\|_2 \leq 1\}$

sum of dyads: $A = \sum_i \sigma_i u_i v_i^T$

U maps $\mathcal{E}(\tilde{\Sigma}) \to \mathcal{E}(A)$

$$\text{Tr}(A^TA) = \text{Tr}(V\tilde{\Sigma}^T U^T U \tilde{\Sigma} V^T) = \text{Tr}(V\tilde{\Sigma}^T\tilde{\Sigma}V^T) = \text{Tr}(\tilde{\Sigma}^2)$$