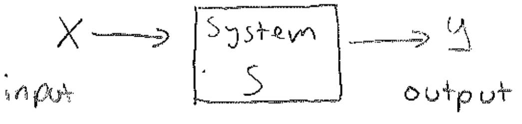
9/2 Lecture

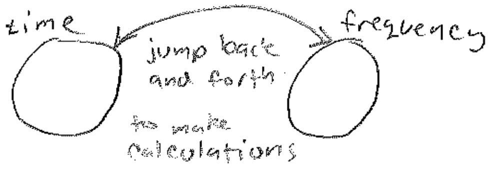
EE 20

2heh, nen, little tail ☺

Systems and Signals



$x(t)$ = value of x at time t



☆ Major IDEA ☆: How to Express signal in its fundamental parts. i.e. break signal into sum of complex exponentials.

☆ Plotting is an Important skill ☆

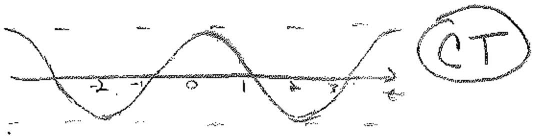
Signals:

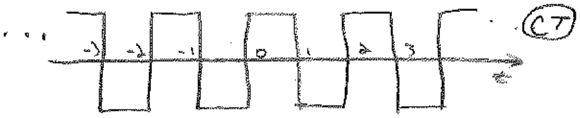
$\underset{\text{domain}}{x:\mathbb{R}} \to \underset{\text{range}}{\mathbb{R}}$

2 types of Signals:

CT: Continuous time

DT: Discrete time





$\mathbb{R} \to \{-1, 1\}$

notation:

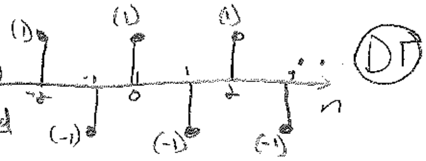
$x \equiv$ signal

$x(n)$: value of the signal x at time n

function ⟺ signal

$x(n) = \cos(\pi n)$

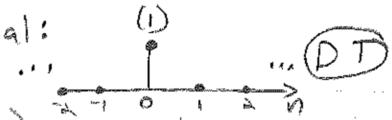
$x(\frac{1}{2})$ = undefined



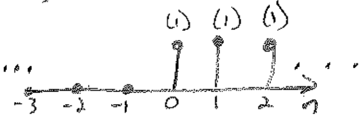
A fundamental Signal:

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

Kronecker delta (Impulse)



$$u(n) = \begin{cases} 0 & n<0 \\ 1 & n\geq 0 \end{cases}$$



unit step function can also be written as a series of δ

$$u(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots + \delta(n-k) = \sum_{k=0}^{\infty} \delta(n-k)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

ie

$$u(n) = \delta(-2) + \delta(-1) + \underset{\uparrow}{\delta(0)} + \delta(1) + \cdots \delta(n)$$

if $n \geq 0$, this term will exist and is always 1

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Review: $\delta(n) \doteq$  = (DT) impulse

LTI: Linear Time Invariant

$x \longrightarrow \boxed{\text{LTI system}} \longrightarrow y$

$\delta(n) \longrightarrow \boxed{\text{LTI system } H} \longrightarrow y(n) = \boxed{h(n)}$ ☆

↑ Impulse response

☆ if we know h, we can determine the output to any system ☆

(DT) unit step:

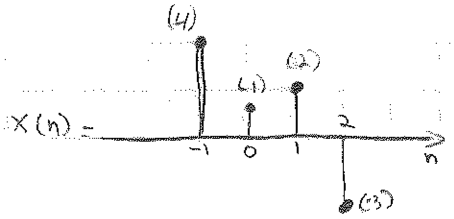
$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

$$= u(0)\delta(n) + u(1)\delta(n-1) + u(2)\delta(n-1) + \cdots$$

$$= \cdots u(-1)\delta(n+1) + u(0)\delta(n) + u(1)\delta(n-1) + \cdots$$

$$= \underline{\sum_{k=-\infty}^{\infty} u(k)\,\delta(n-k) = \sum_{k=0}^{\infty} \delta(n-k)}$$

→ useful "identity" to notice to simplify problems

$x(n) =$ 

→ Decompose as a linear combo of shifted impulse.

$4\delta(n+1) + \delta(n) + 2\delta(n-1) - 3\delta(n-2)$: **WRONG** ☆

☆ Why?! Because x is a function so it is the value of x at that point.

RIGHT: $x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2)$

☆ Any arbitrary signal x can be decomposed into a linear combo of shifted impulses ☆

LTI System Overview (Rapid-Fire View)

$\overset{\text{impulse}}{\delta(n)} \longrightarrow \boxed{H} \longrightarrow \overset{\text{impulse response}}{h(n)}$

$\overset{\text{"scaling factor"}}{x(0)}\overset{\text{impulse}}{\delta(n)} \longrightarrow \boxed{H} \longrightarrow \overset{\text{"scaled" impulse response}}{x(0)\,h(n)}$

$q(n) \longrightarrow \boxed{H} \longrightarrow r(n)$

$q(n) + \delta(n) \longrightarrow \boxed{H} \longrightarrow \overbrace{r(n) + h(n)}^{\text{additivity}}$

☆ for a linear system you need ☆

1- linear scaling

2- additivity

LINEAR

$x_1 \longrightarrow \overset{\text{Linear}}{\boxed{H}} \longrightarrow y_2$

$x_2 \longrightarrow \overset{\text{Linear}}{\boxed{H}} \longrightarrow y_2$

$\alpha_1 x_1 + \alpha_2 x_2 \longrightarrow \overset{\text{Linear}}{\boxed{H}} \longrightarrow \alpha_1 y_1 + \alpha_2 y_2$

☆ must be true for ALL $x \in \underset{\text{scalar}}{\alpha}$

Time-Invariance

$x \longrightarrow \boxed{H} \longrightarrow y$

$\hat{x} \longrightarrow \boxed{H} \longrightarrow \hat{y}$

$\hat{x} = x(n-N)$

$\forall n, \forall N \in \mathbb{Z}$

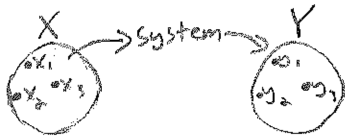
We say a system is $\overbrace{\text{Time-Invariant}}^{\text{TI}}$ if

$\hat{y} = y(n-N) \quad \forall n$

☆ this must be true for ALL VALID input x

9/4 - Lecture

$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$



☆ A system maps one value to another ☆ (system → function)

Difference of Signals vs Systems

Signals : $X : \underset{\text{domain}}{\mathbb{Z}} \rightarrow \underset{\text{range}}{\mathbb{R}}$

Systems : $H : \underset{\text{domain}}{X} \longrightarrow \underset{\text{range}}{Y}$

set of signals → set of signals

LTI system recap

$$x \rightarrow \boxed{\overset{LTI}{H}} \rightarrow y$$

$$\underset{\forall n}{x(n) = \delta(n)} \rightarrow \boxed{\overset{LTI}{H}} \rightarrow y(n) = h(n)$$

☆ By convention ☆

$$\hat{x}(n) = \delta(n-1) \rightarrow \boxed{\overset{LTI}{H}} \rightarrow y(n-1) = \overbrace{h(n-1)}^{\text{shifted in time b/c T-I}}$$

$$\hat{x}(n) = \alpha\delta(n-1) \rightarrow \boxed{\overset{LTI}{H}} \rightarrow \alpha y(n-1) = \overbrace{\alpha h(n-1)}^{\text{scaled \& shifted b/c L \& TI}}$$

what if...

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \longrightarrow \boxed{\overset{LTI}{H}} \rightarrow y = ?$$

$$x(n) \longrightarrow \boxed{\overset{LTI}{H}} \rightarrow y(n) = \underbrace{\sum_{k=-\infty}^{\infty} x(k)h(n-k)}_{y = x * h}$$

why?: CONVOLUTION

Convolution: ☆ to roll together ☆ → verb → action

$y = x ☆ h$ (or $x * h$)

$$y(n) = \underbrace{\sum_{\ell=-\infty}^{\infty} h(\ell)\, x(n-\ell)}_{y = h * x}$$

$h*x = x*h$ → just do a change of variable and it works out.

let $u = n - \ell$

$\ell = n - u$

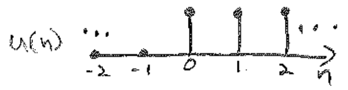
$$y(n) = \sum_{\ell=-\infty}^{\infty} h(\ell)x(n-\ell)$$

$$\sum_{u=-\infty}^{\infty} h(n-u)\, x(\ell)$$

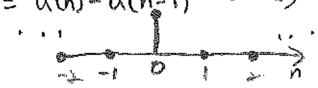
$(\therefore)\ y(n) = h*x = x*h$

$$u(n) \longrightarrow \boxed{\overset{LTI}{H}} \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} u(k)\, h(n-k)$$

$$\sum_{k=-\infty}^{\infty} u(k)\delta(n-k) \longrightarrow \boxed{\overset{LTI}{H}} \longrightarrow y(n) = \sum_{k=0}^{\infty} h(n-k)$$

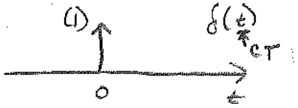


$\delta(n) = u(n) - u(n-1) \longrightarrow \delta(n) = \frac{u(n) - u(n-1)}{1}$ ← normalized



☆ Dirac Delta idealizes a function thats large over a tiny interval and negligible outside ☆

CT impulse :

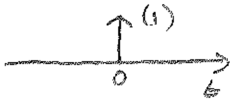


$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

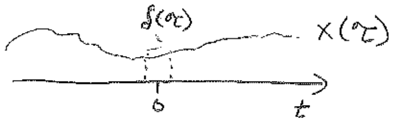
9/9 – Lecture

Dirac Delta continued

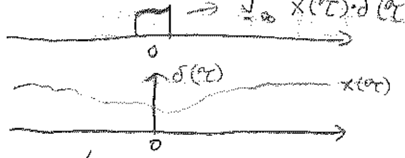
$$\underbrace{\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}}_{\text{pretty much meaningless}}$$



$$= \underbrace{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}_{\text{normalization}}$$



if you multiply $x(\tau) \cdot \delta(\tau)$



$$\rightarrow \int_{-\infty}^{\infty} x(\tau) \cdot \delta(\tau)\,d\tau \approx x(0)$$

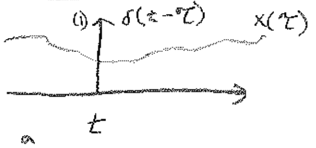
$$\int_{-\infty}^{\infty} x(\tau) \cdot \delta(\tau) \cdot d\tau = \boxed{x(0)}$$

ONLY has value when $\tau = 0$ $(\therefore)$

Pretty much by definition

☆ You can also scale the dirac delta to scale the signal ☆

Sifting property



$$(\therefore) \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

only non zero when $\tau = t$

$(\therefore) = x(t)$

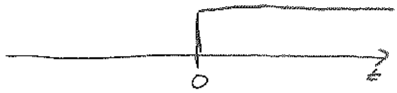
$$\underbrace{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

decomposition of Ⓒⓣ signal x in terms of shifted impulses

$$\text{CT} \longleftrightarrow \text{counterpart} \longleftrightarrow \text{DT}$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t) \Longleftrightarrow x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

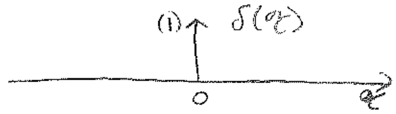
CT Unit Step | u(t)



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

★ sometimes it is convienient

$$u(t) = \begin{cases} 1 & t > 0 \\ \tfrac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$ ★



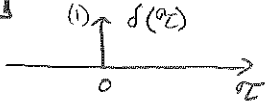
$$u(t): \int_{0}^{\infty} \delta(t-\tau)\,d\tau \quad \text{CT}$$

$\updownarrow$ counter part $\updownarrow$

$$u(n): \sum_{k=0}^{\infty} \delta(n-k) \quad \text{DT}$$

Example: Impulse

$$\frac{u(t) - u(t-1)}{1} \quad 1$$



$$u(t) \int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} = u(t)$$

$$du = \delta(\tau)\Big|_{-\infty}^{t} = \delta(t)$$

CT LTI Systems

Linear

$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$

$$x = \alpha_1 x_1 + \alpha_2 x_2 \rightarrow \boxed{H} \rightarrow y = \alpha_1 y_1 + \alpha_2 y_2$$

$\forall x_1, x_2 \in X \quad \forall$ scalars α_1, α_2

L

$$x_1 \rightarrow \boxed{H} \rightarrow y_1$$

L

$$x_2 \rightarrow \boxed{H} \rightarrow y_2$$

Time Invariance

TI

$$x \rightarrow \boxed{H} \rightarrow y$$

TI

$$\hat{x} \rightarrow \boxed{H} \rightarrow \hat{y}$$

if $\forall x \in X$ & $\forall T \in \mathbb{R}$

we have $\hat{y}(t) = y(t-T)$ $\forall t$

then we say H is TI

where $\hat{x}(t) = x(t-T)$

$T \in \mathbb{R}$

☆ An LTI system satisfies both superposition and time invariance ☆

$$\delta(t) \longrightarrow \boxed{H} \rightarrow h(t) \rightarrow \text{The impulse response}$$

$$\delta(t-\tau) \rightarrow \boxed{H} \rightarrow h(t-\tau) \rightarrow \text{because of time invariance}$$

$$x(t)\,\delta(t-\tau) \rightarrow \boxed{H} \rightarrow h(t-\tau)\,x(t) \rightarrow \text{because of scaling property of Linearity}$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{H} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \rightarrow \text{convolution integral (CT)}$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \longrightarrow \boxed{H} \rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \rightarrow \text{convolution sum (DT)}$$

9/10 — Discussion

(5)

Ex. 1 $y(n) = \frac{x(n) + x(n-1)}{2}$

a) is system linear?

b) is system time invariant?

c) if LTI then find and plot $h(n)$

a) $\alpha_1 x_1 + \alpha_2 x_2 \longrightarrow \boxed{H} \longrightarrow y(n) = \frac{\alpha_1 x_1(n) + \alpha_1 x_1(n-1) + \alpha_2 x_2(n) + \alpha_2 x_2(n-1)}{2}$

$\frac{\alpha_1 x_1(n) + \alpha_1 x_1(n-1)}{2} + \frac{\alpha_2 x_2(n) + \alpha_2 x_2(n-1)}{2}$

$= \frac{\alpha_1(x_1(n) + x_2(n-1))}{2} + \frac{\alpha_2(x_2(n) + x_2(n-1))}{2}$

(∴) Linear $\overset{\checkmark}{=} \alpha_1 y_1 + \alpha_2 y_2$ ...

b) TI?

$\hat{x}(n) \longrightarrow \boxed{H} \longrightarrow \hat{y}(n)$

rewrite y as $\hat{y}$ by adding hat to $x \to \hat{x}$, then sub $\hat{x}$ expression for x

★ always defined as this ★

let $\hat{x}(n) = x(n-N)$

(∴) $\hat{x}(n) \longrightarrow \boxed{H} \longrightarrow \hat{y}(n) = \frac{\hat{x}(n) + \hat{x}(n-1)}{2}$

plug this into $\hat{x}$ def to get

$= \frac{x(n-N) + x(n-1-N)}{2}$

(∴) TI ✓ $\hat{y}(n) = y(n-N)$

then check to make sure $\hat{y} = y$

c) find and plot $h(n)$.

↳ do later after Lecture catches up with this...

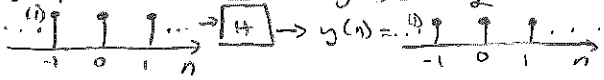
Ex 2. Determine System output for the following input signals (using Ex 1)

a) $x_1(n) = 1$

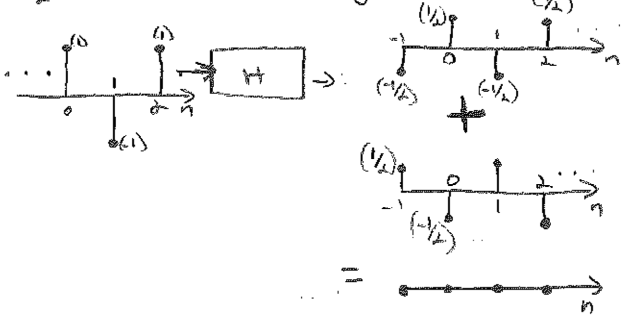
b) $x_2(n) = \cos(\pi n)$

c) $x_3(n) = u(n)$

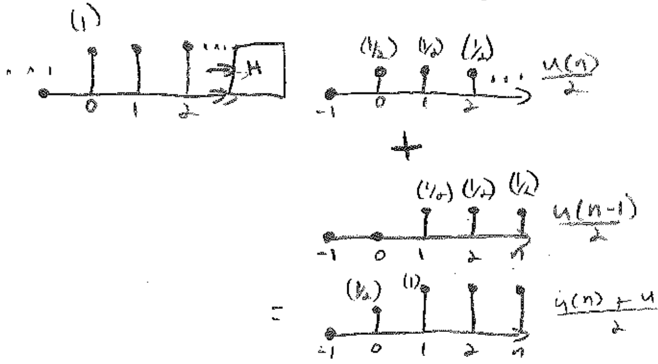
a) $x_1(n) = 1 \longrightarrow \boxed{H} \longrightarrow y(n) = \frac{1+1}{2} = \boxed{1}$



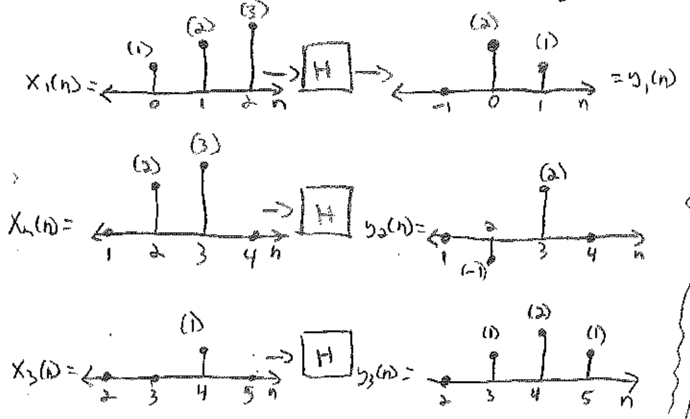
b) $x_2(n) = \cos(\pi n) \longrightarrow \boxed{H} \longrightarrow y(n) = \frac{\cos(\pi n) + \cos(\pi n - \pi)}{2} (\frac{1}{2}) = \boxed{0}$



c) $x_3(n) = u(n) \longrightarrow \boxed{H} \longrightarrow y(n) = \begin{cases} 0 & y < 0 \\ 1/2 & y = 0 \\ 1 & y > 0 \end{cases}$ by visual below



Ex 3. Consider a Time-Invariant System H. given the following input/output pairs is H also Linear?

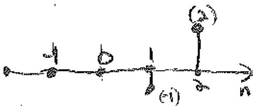


can I create one of the x signals by shifting and scaling the others ie linear combination?

→ Yes. $x_1(n) = x_2(n+1) + x_3(n+4)$

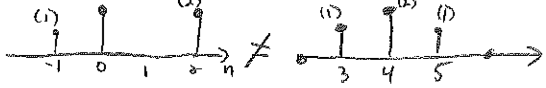
then $y_1(n) = y_2(n+1) + y_3(n+4)$

$y_2(n+1) =$



$y_3(n+4) =$

(∴) $y_1(n) =$



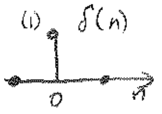
(∴) not linear.

9/11 - Lecture

Review:

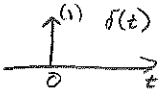
4 signals: 2 in DT
2 in CT

DT



$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

CT



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

Frequency

$x(t) = e^{i\omega t}$

generalize to complex exponentials

$z = a+bi \rightarrow e^{a+bi} = e^{a} e^{bi}$

☆ $i = e^{i\pi/2}$ ☆

Example:

$x(t) = e^{i2\pi t}$

↳ 1 Hz phasor. completes one revolution per second.

in general: $e^{i\omega t} \rightarrow \omega = 2\pi f$

$\underset{\text{freq in rad/s}}{\uparrow}$ $\underset{\text{frequency in Hz}}{\uparrow}$

☆ Frequencies can be negative or positive ☆

$$\boxed{\begin{array}{l} \text{if ccw} \rightarrow + \Rightarrow \omega > 0 \rightarrow \text{ccw} \\ \text{if cw} \rightarrow - \Rightarrow \omega < 0 \rightarrow \text{cw} \end{array}}$$

☆ Know this ish well! ☆

Euler's Formula

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$e^{-i\omega t} = \cos(-\omega t) + i\sin(-\omega t) = \underset{f(x)=f(-x)}{\overset{\text{even func}}{\overset{\downarrow}{\cos(\omega t)}}} - i\,\underset{f(-x)=-f(x)}{\overset{\text{odd func}}{\overset{\downarrow}{\sin(\omega t)}}}$$

$$e^{i\omega t} + e^{-i\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

9/16 - Lecture

Complex Exponentials

☆ Recap: learn Euler's Formula☆

$x(t) = \sin(\omega t) \approx$

$x(t+p) = x(t)\ \forall t \rightarrow$ smallest p is the $\boxed{\text{Fundamental}}$ period

$\sin(\omega(t+p)) = \sin(\omega t + \omega p) \qquad \omega p = 2\pi \underset{\uparrow}{k}$

$= \sin(\omega t + 2\pi)$ pick smallest positive

$= \sin(\omega t)$ k :=> k = 1

$\omega p = 2\pi \Rightarrow p = \frac{2\pi}{\omega}$

☆ CT => $\underline{\underline{\text{NO}}}$ highest frequency ☆

in DT: $x(n) = e^{i\omega n}$

$g(n) = e^{i(\omega+2\pi)n} = e^{i\omega n + i2\pi n}$

$= e^{i\omega n} \qquad = e^{i\omega n}\cancelto{1}{e^{i2\pi n}}$

$e^{i2\pi n} = \cancelto{1}{\cos(2\pi n)} + i\cancelto{0}{\sin(2\pi n)}$

$(\therefore)\ e^{i2\pi n} = 1$

☆ DT: sin are periodic W.R.T the freq.☆

$\sin(\omega n), \cos(\omega n), e^{i\omega n}$, 2π = period for all of these.

☆DT: sins are not necessarily periodic in $\underline{\underline{n}}$☆

$x(n+p) = x(n) \quad \forall n \in \mathbb{Z} \quad \exists p \in \mathbb{Z}$

whats the period of $x(n) = \sin(n)$?

if x is periodic, there must be an <u>integer</u>

p s.t. $\sin(n+p) = \sin(n)\ \forall n$

$p = 2\pi k$ is $\underline{\underline{\text{not}}}$ an integer

$(\therefore)$ $x(n) = \sin(n)$ cannot be peridoic in n

Ex: $x(n) = \sin(\omega n) \rightarrow$ want this to be periodic

$\sin(\omega(n+p)) = \sin(\omega n + \omega p) \qquad \omega p = 2\pi k$

$= \sin(\omega n) \qquad (\therefore)\ p = \frac{2\pi k}{\omega}$

$\left.\begin{matrix}\sin(\omega n)\\ \cos(\omega n)\\ e^{i\omega n}\end{matrix}\right\}$ are all periodic $\underline{\underline{\text{iff}}}$ ω is a rational multiple of π

i.e. $\omega = \frac{l}{m}\pi \quad l, m \in \mathbb{Z},\ m \neq 0$

DT frequencies

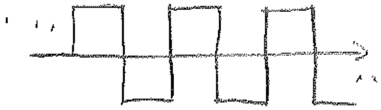
$x(n) = 1\ \forall n \longrightarrow$

$x(n) = e^{i2\pi k n}$

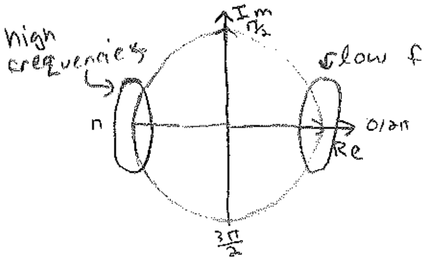
$= e^{i0n} \rightarrow \omega = 0 \rightarrow$

$= e^{i2\pi n} \rightarrow \omega = 2\pi$ ↗ ↑ even multiples of π

☆ the fastest frequency in DT: odd multiples of π ☆



signal $x(n) = (-1)^n = e^{i\pi n} = e^{i3\pi n} = e^{i(\pi+2k\pi)n} = \cos(\pi n)$



why do we care about complex exponentials?

DT-LTI systems and complex exponentials

$x \longrightarrow \boxed{H} \longrightarrow y$

$y(n) = \sum_{l=-\infty}^{\infty} x(l)h(n-l) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

$y = x * h = h * x$ let $k = n - l$

if $x(n) = e^{i\omega n} \rightarrow$ determine y(n).

$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{i\omega(n-k)} \rightarrow e^{i\omega n}\underbrace{\sum_k h(k)e^{-i\omega k}}$

H(ω) - Frequency response of the LTI system

same

$e^{i\omega n} \longrightarrow \boxed{\begin{matrix}h(n)\\ H(\omega)\end{matrix}} \longrightarrow \underbrace{H(\omega)}_{\text{scaling factor}}\ \underbrace{e^{i\omega n}}_{\text{original signal}}$

For linear Algebra:
Eigen vectors $Av = \lambda v$

Observations

☆Any $\underline{\underline{\text{LTI}}}$ system $\underline{\text{cannot}}$ produce new frequencies☆

Ex. $x(n) \longrightarrow \overset{\text{non LTI}}{\boxed{(\cdot)^2}} \rightarrow y(n) = x^2(n) \Rightarrow x(n) = e^{i\frac{\pi}{3}n} \rightarrow y(n) = e^{i\frac{2\pi}{3}n}$

Complex exponentials are eigen functions of LTI systems.

if a complex exponential is applied, a <u>scaled version</u> of the <u>SAME</u> complex exp. comes out.

$(\therefore)\ H(\omega) = \sum_k h(k)e^{-i\omega k}$

$H(\omega+2\pi) = \sum_k h(k)e^{-i(\omega+2\pi)k} = \sum_k h(k)e^{-i\omega k}\cancelto{1}{e^{-i2\pi k}}$

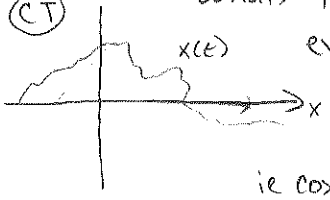
$H(\omega+2\pi) = H(\omega)$

9/17 – Discussion

time domain ⟶ frequency domain

whats the intuition?

Ⓒⓣ



every signal can be broken down.

$$X(t) = \sum A_0(w_0)\, e^{iwt}$$

$$\text{ie } \cos(wt) = \frac{e^{iwt} + e^{-iwt}}{2}$$

magnitude
|A|

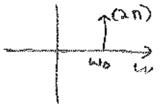
phase ∡A

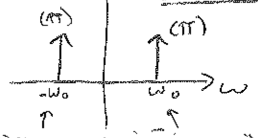
☆just an example, NOT a real cos(wt) plot☆

Plot cos(wt) mapped to the frequency domain

$$e^{iw_0t} \longrightarrow 2\pi(w - w_0)$$

↑ fourier Transform (learn about later)





$$\frac{2\pi}{2}\delta(w - w_0) + \frac{2\pi}{2}\delta(w - -w_0)$$

$$\cos(wt) = \frac{1}{2}e^{iwt} + \frac{1}{2}e^{-iwt}$$

Spectrum – plot of the frequency domain representation.

plot spectrum of $x(t) = \cos(2\pi(101)t) + \cos(2\pi(99)t)$

$$= \frac{e^{i2\pi(101)t} + e^{-i2\pi(101)t}}{2} + \frac{e^{i2\pi(99)t} + e^{-i2\pi(99)t}}{2}$$

$$= \frac{1}{2}\left(e^{i2\pi 101t} + e^{i2\pi 99t}\right) + \frac{1}{2}\left(e^{-i2\pi 101t} + e^{-i2\pi 99t}\right)$$

$$= \frac{1}{2}e^{i2\pi 100t}\left(e^{i2\pi t} + e^{-i2\pi t}\right) + \frac{1}{2}e^{-2\pi 100t}\left(e^{-i2\pi t} + e^{i2\pi t}\right)$$

$$= \underbrace{2\cos(2\pi t)}\cdot\underbrace{\cos(2\pi 100t)}$$

used in a lot of things:

music, amplitude modulation-

radio transmission

⑧

Convolving

recap:

$$\delta(n) \rightarrow \boxed{H} \rightarrow h(n) \leftarrow \text{impulse response}$$

$$\underset{\substack{\uparrow \\ \text{same frequency}}}{e^{iw_0n}} \longrightarrow \boxed{H} \rightarrow H(w_0)\underset{\substack{\uparrow \\ \text{same frequency}}}{e^{iw_0n}}$$

Frequency response at w_0

2 point moving averager

$$\rightarrow \boxed{H} \rightarrow y(n) = \frac{x(n) + x(n-1)}{2}$$

in frequency domain

$$e^{iwn} \longrightarrow \boxed{H} \longrightarrow \frac{e^{iwn} + e^{iw(n-1)}}{2}$$

$$H(w)\cancel{e^{iwn}} = \frac{\cancel{e^{iwn}} + \cancel{e^{iwn}}e^{-iw}}{2}$$

$$H(w) = \frac{1 + e^{-iw}}{2}$$

$$|H(w)| = \sqrt{a^2 + b^2}$$

☆cool trick☆ ⇐ very useful

factor out $e^{-\frac{iw}{2}}$

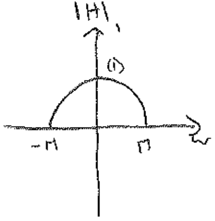
$$H(w) = e^{-\frac{iw}{2}}\left(\frac{e^{\frac{iw}{2}} + e^{-\frac{iw}{2}}}{2}\right)$$

$$H(w) = e^{-i\frac{w}{2}}\,\underbrace{\cos\left(\frac{w}{2}\right)}_{\text{Magnitude}}$$

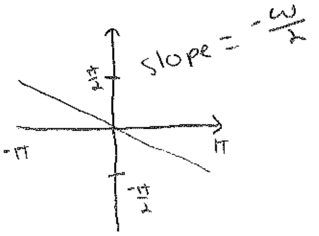
$e^{-i\left(\frac{w}{2}\right)}$ = period

i · period = phase = $-\frac{w}{2}$

magnitude
|H|



phase



9/18 - Lecture

Review Complex Exponentials thru DT-LTI

$$x(n)=e^{i\omega n} \rightarrow \boxed{H} \rightarrow \underbrace{H(\omega)}_{\text{Frequency Response}} e^{-i\omega n}$$

Fourier Transform

$$h(n) \xleftrightarrow[\text{Learn later}]{\mathcal{F}} H(\omega) = \sum_n h(n) e^{-i\omega n}$$

Basic Filters

two point moving Averager filter

$$x(n) \rightarrow \boxed{H} \rightarrow y(n) = \frac{x(n)+x(n-1)}{2}$$

Impulse response? why? make sure it is LTI first!

$$\hat{x}(n) = x(n-N) \rightarrow \boxed{H} \rightarrow \hat{y}(n) = \frac{\hat{x}(n-N)+\hat{x}(n-N-1)}{2}$$

$$y(n-N) = \frac{x(n-N)+x(n-N-1)}{2}$$

= same ✓ → T-I

So H is LTI!

$$h(n) = \frac{\underbrace{\delta(n)}_{n=0} + \underbrace{\delta(n-1)}_{n=1}}{2} =$$

$$\hookrightarrow H(\omega) = \sum_n h(n) e^{-i\omega n}$$
$$= h(0)\overset{1}{e^{-i\omega 0}} + h(1)e^{-i\omega}$$
$$= h(0) + h(1)e^{-i\omega} = \frac{1+e^{-i\omega}}{2}$$

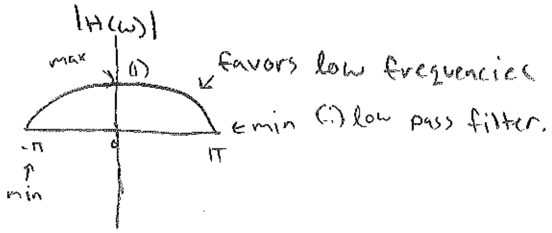
To get insight into the filter's behavior we plot $|H(\omega)|$ & $\measuredangle H(\omega)$

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{magnitude response}} e^{i\underbrace{\measuredangle H(\omega)}_{\text{phase response}}}$$

To get magnitude response, balance the exponents, using cool trick.

$$H(\omega) = \left(\frac{e^{i\omega/2}+e^{-i\omega/2}}{2}\right)e^{-i\omega/2}$$

$$H(\omega) = \cos\left(\frac{\omega}{2}\right)e^{-i\frac{\omega}{2}}$$

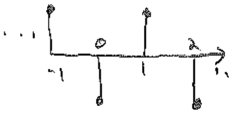


(∴) low pass filter

frequency vs time domain

$$x(n) = 1 \ \forall n \rightarrow \boxed{H} \rightarrow y(n) = 1$$

$$x(n) = e^{i0n} \rightarrow \boxed{H} \rightarrow H(\omega) = \cos\left(\frac{\omega}{2}\right)e^{-i\frac{\omega}{2}} \rightarrow \underbrace{H(0)}_{1}\underbrace{e^{i0n}}_{1} = 1$$



$$\rightarrow \boxed{H} \rightarrow y(n) = \frac{1+(-1)}{2} = 0$$

$$x(n) = e^{i\pi n} \rightarrow \boxed{H} \rightarrow y(n) = \underbrace{H(\pi)}_{0}\underbrace{e^{i\pi n}}_{1} = 0$$

what about a two point moving differencing filter?

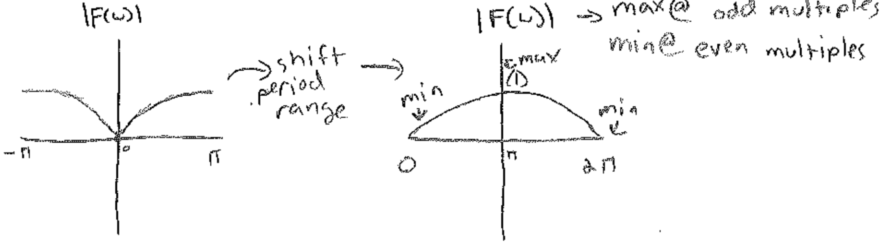
$$y(n) = \frac{x(n)-x(n-1)}{2}$$

$$F(\omega) = \frac{1-e^{-i\frac{\omega}{2}}}{2} = \frac{e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}e^{i\frac{\omega}{2}}}{2} = i\sin\left(\frac{\omega}{2}\right)e^{-i\frac{\omega}{2}}$$

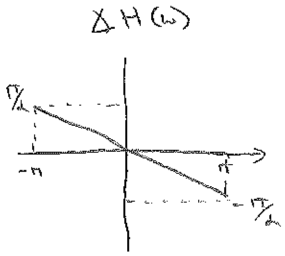
$$i = e^{i\frac{\pi}{2}}$$

$$|F(\omega)| = \left|\sin\left(\frac{\omega}{2}\right)\right|$$

$$F(\omega) = \sin\left(\frac{\omega}{2}\right)e^{-i\left(\frac{\omega}{2}-\frac{\pi}{2}\right)}$$



(∴) High Pass filter



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9-25 - Lecture

Frequency response of DT-LTI System

$y(n) = \alpha y(n-1) + x(n)$

linear constant coeffecient difference equation (LCCDE)

$\underbrace{y(n) = 0 \quad n<0}_{\text{initially at rest}}$ $\qquad x \rightarrow \boxed{H} \rightarrow y$

$|\alpha| < 1$

whats the impulse response h(n)?

know $h(n) = 0 \quad \forall\, n<0$ (initially at rest)

$h(n) = \alpha h(n-1) + \delta(n)$

$h(0) = \alpha \cancel{h(0-1)}^{0} + 1 = 1$

$h(1) = \alpha h(0) + \cancel{\delta(1)}^{0} = \alpha$

$h(2) = \alpha h(1) + \cancel{\delta(2)}^{0} = \alpha^2$

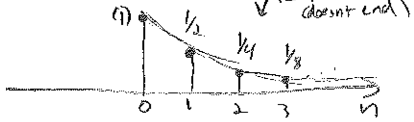
$h(3) = \alpha(h(2)) = \alpha^3$

$\vdots$

$\therefore h(n) = \alpha^n \underbrace{u(n)}$

nonzero $n \geq 0$

let $\alpha = \frac{1}{2}$ → plot



whats the frequency response H(ω)?

$\boxed{\text{method 1:}}$

let $x(n) = e^{i\omega n} \rightarrow y(n) = H(\omega)e^{i\omega n}$

need $y(n-1) = H(\omega)e^{i\omega(n-1)}$

plug into the input/output LCCDE and solve for H(ω)

$H(\omega)\cancel{e^{i\omega n}} = \alpha H(\omega)\cancel{e^{i\omega(n-1)}} + \cancel{e^{i\omega n}}$

$H(\omega) = \alpha H(\omega)e^{-i\omega} + 1$

$H(\omega) - \alpha H(\omega)e^{-i\omega} = 1$

$H(\omega)(1-\alpha e^{-i\omega}) = 1$

$$\boxed{H(\omega) = \frac{1}{1-\alpha e^{-i\omega}}}$$

(IIR) ↑

Infinite Duration Impulse Response

Method 2:

$$H(\omega) = \sum_{n=0}^{\infty} h(n)e^{-i\omega n}$$

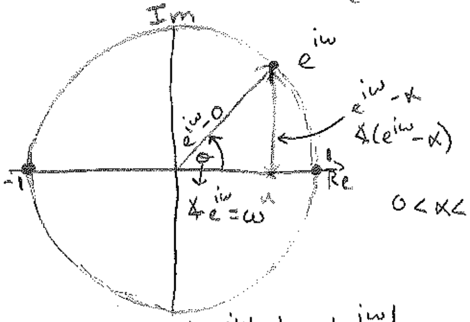
$$H(\omega) = \sum_{n=0}^{\infty} (\alpha^n)e^{-i\omega n} = \overbrace{\sum_{n=0}^{\infty}(\alpha e^{-i\omega})^n}^{\text{geometric series}} = \frac{1}{1-\alpha e^{-i\omega}}$$

$|\alpha e^{-i\omega}| = |\alpha|\cancel{|e^{-i\omega}|}^{1}$

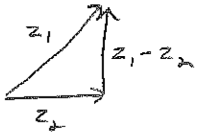
if $|\alpha| < 1$ →

Plot H(ω)!

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}} = \frac{e^{i\omega}}{e^{i\omega}-\alpha} = \frac{e^{i\omega}-0}{e^{i\omega}-\alpha}$$



$0 < \alpha < 1$

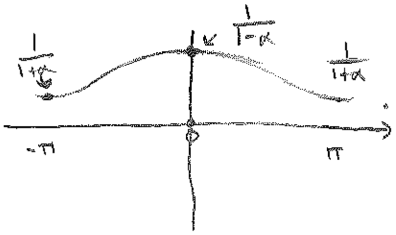


$$|H(\omega)| = \left|\frac{e^{i\omega}}{e^{i\omega}-\alpha}\right| = \frac{|e^{i\omega}|}{|e^{i\omega}-\alpha|} = \frac{1}{|e^{i\omega}-\alpha|} = \frac{|\nearrow|}{|\nearrow|} = \frac{1}{|\nearrow|}$$

$|\nearrow|$ is minimum at $\omega = 0$

maximum at $\omega = \pi$

$\boxed{\therefore \text{ low pass filter.}}$



$H(0) = \frac{1}{1-\alpha}$

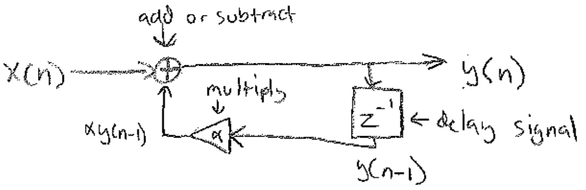
$H(\pi) = \frac{1}{|-1-\alpha|} = \frac{1}{1+\alpha}$

☆ sharper peak → move α closer to the unit circle ☆

how to turn ↑ into a high pass filter?

move α → $-1 < \alpha < 0$

Delay-Adder-Gain (DAG) Block Diagram



$\boxed{z^{-1}}$ is a unit-delay block

$q(n) \rightarrow \boxed{z^{-1}} \rightarrow q(n-1)$

This is a first order filter, b/c you need a minimum of one delay element

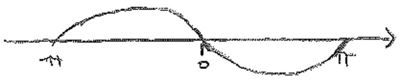
what about the phase response $\measuredangle H(\omega)$?

$$H(\omega) = \frac{e^{i\omega}}{e^{i\omega}-\alpha}$$

$$\measuredangle H(\omega) = \underset{\text{top}}{\measuredangle e^{i\omega}} - \underset{\text{bottom}}{\measuredangle(e^{i\omega}-\alpha)}$$

$$= \measuredangle \nearrow - \measuredangle \nearrow = \omega - \measuredangle(e^{i\omega}-\alpha)$$

$\measuredangle\, \omega = 0 = -$ b/c cw



9/30 - Lecture

CT - LTI Systems

$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$

impulse   impuse response

$\delta(t) \longrightarrow \boxed{H} \longrightarrow y(t) = h(t)$

shifted

$\delta(t-\tau) \longrightarrow \boxed{H} \longrightarrow y(t) = h(t-\tau)$

shifted and scaled

$x(\tau)\delta(t-\tau) \rightarrow \boxed{H} \rightarrow y(t) = x(\tau)h(t-\tau)$

$$\underbrace{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau}_{\text{Sifting}} \rightarrow \boxed{H} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,dt$$

$$y(t) = (x \star h)(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

let $\lambda = t - \tau \rightarrow d\lambda = -d\tau$

$\boxed{\tau = t - \lambda}$ evaluate new limits   flip limits of integration flip sign too

$$= \int_{\infty}^{-\infty} h(\lambda)\, x(t-\lambda)(-d\lambda)$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\,d\lambda = (h \star x)(t)$$

$(\therefore)\ x \star h = h \star x$

$x \rightarrow \boxed{H} \rightarrow y$

$h \rightarrow \boxed{X} \rightarrow y$

Frequency response of CT-LTI systems

$x(t) = e^{i\omega t} \longrightarrow \boxed{H} \rightarrow y(t) = ?$

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\,x(t-\lambda)\,d\lambda = \int_{-\infty}^{\infty} h(\lambda)\,\underbrace{e^{i\omega(t-\lambda)}}_{e^{i\omega t}e^{-i\omega\lambda}}\,d\lambda$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} h(\lambda)\, e^{-i\omega\lambda}\,d\lambda\right)}_{\text{function of } \omega \,=\, H(\omega)} e^{i\omega t}$$

$$H(\omega) = \int_{-\infty}^{\infty} \overset{\uparrow}{h(\lambda)}\, e^{-i\omega\lambda}\,d\lambda \;\Big\}\; \text{Frequency response of CT-LTI system}$$

dummy variable

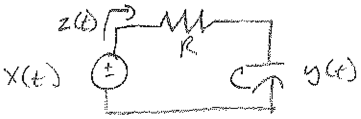
↓

similar to the DT version

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$$

☆ the CT version is not periodic, in general, in the freq variable ω. b/c ($e^{i\omega t}$ is not periodic in ω) ☆

Example:



$V = IR \rightarrow I = \frac{V}{R}$

$$z(\tau) = \frac{x(\tau) - y(\tau)}{R}$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} z(\tau)\,d\tau$$

$y(t) = 0\ \forall t < 0$

$$y(t) = \frac{1}{RC}\int_{-\infty}^{t} [x(\tau) - y(\tau)]\,d\tau$$

$\frac{d}{dt}$ ↓

$(\therefore)\ \dot{y}(t) = \frac{1}{RC}[x(t) - y(t)]$

$RC\dot{y}(t) = x(t) - y(t) \leftarrow \boxed{\text{LCCDE}}$

want to know $H(\omega)$, the freq. response of the circuit.

Let $x(t) = e^{i\omega t} \rightarrow y(t) = H(\omega)e^{i\omega t}$

$(\therefore)\ \dot{y}(t) = i\omega H(\omega)e^{i\omega t}$

$RC(i\omega H(\omega))\cancel{e^{i\omega t}} = \cancel{e^{i\omega t}} - H(\omega)\cancel{e^{i\omega t}}$

$Rci\omega H(\omega) = 1 - H(\omega)$

$H(\omega)[Rci\omega + 1] = 1$

$H(\omega) = \frac{1}{Rci\omega + 1}$

$\measuredangle H(\omega) = \measuredangle \frac{1}{i\omega RC + 1} = \measuredangle 1 - \measuredangle(i\omega RC + 1)$

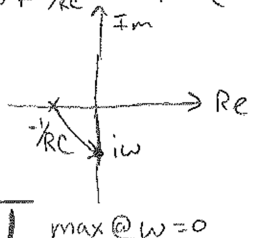
$= 0 - \tan^{-1}(i\omega RC)$

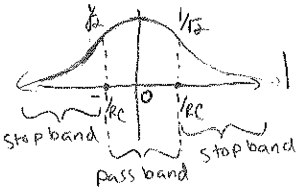
$\measuredangle H(\omega) = -\tan^{-1}(\omega RC)$

$|H(\omega)|$?

$H(\omega) = \frac{1}{i\omega RC + 1} = \frac{1/RC}{i\omega + 1/RC} \rightarrow \frac{1/RC}{i\omega - (-1/RC)}$

$|H(\omega)| = \frac{1/RC}{|i\omega - (-1/RC)|}$





$|H(\omega)| = \frac{\frac{1}{RC}}{|\nearrow|}$

max @ $\omega = 0$

min @ $\omega = \infty$

low pass filter.

System Properties:

linearity

Time invariance

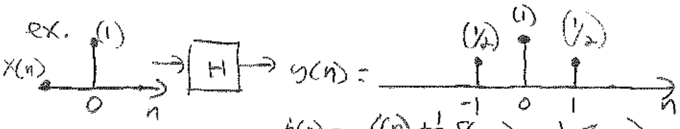
Causality →

memorylessness

we say a system is causal if it doesn't have to look ahead in the input to determine the current output value

Bounded input Bounded output - BIBO

ex.



$y(n) = \delta(n) + \frac{1}{2}\underset{\uparrow}{\delta(n+1)} + \frac{1}{2}\delta(n-1)$

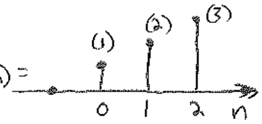
future value

Causal?: $\boxed{\text{NO}}$

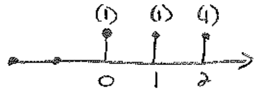
10/1 – Discussion

"Flip and shift" in DT

$x(n) =$



$h(n) =$

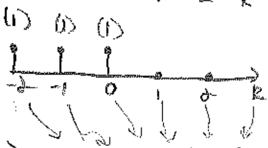


$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$y(0) = \sum_{k} x(k)\,h(-k)$$

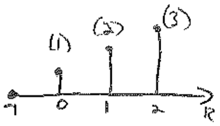
$x(k) =$

$h(-k)$



$\therefore\ y(0) = 0+0+1+0+0+0\ldots = 1$

$X(k) =$



$h(1-k) =$

$y(1) = 0+1+2+0+0\ldots = 3$

$h(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

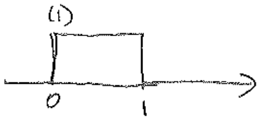
$y(n) = x(n) + x(n-1) + x(n-2)$

$\therefore\ y(0) = x(\cancel{0})^{1} + x(\cancel{-1})^{0} + x(\cancel{-2})^{0} \overset{\checkmark}{=} 1$

$y(1) = x(\cancel{1})^{2} + x(\cancel{0})^{1} + x(\cancel{-1})^{0} \overset{\checkmark}{=} 3$

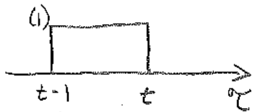
CT

$x(t) * x(t)$ $\qquad x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$



$$= \int_{-\infty}^{\infty} x(\tau)\,x(t-\tau)\,d\tau$$

$x(t-\tau)$



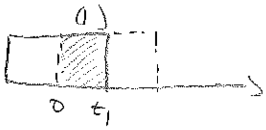
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,x(t-\tau)\,d\tau$$

↑ ↑ this is fliped because the original limits are 0 & 1 → plug 0 into original equation → t-0

plug 1 into $x(t-\tau) \Rightarrow t-1$

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\,x(0-\tau)\,d\tau$$

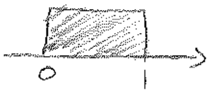
$x(\tau) =$



$y(t_1) = t_1$

$x(0-\tau) = x(-\tau)$

$y(1) = 1$



$y(t_2) = 1 - t_2$

$$y(t) = \begin{cases} 0 & t<0 \\ t & 0<t<1 \\ 1-t & 1<2 \\ 0 & t>2 \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,x(t-\tau)\,d\tau$$

$$= \int_{0}^{1} x(t-\tau)\,d\tau$$

$$= -\int_{t}^{t-1} x(s)\,ds \qquad \begin{aligned} s &= t-\tau \\ ds &= -d\tau \\ d\tau &= -ds \end{aligned}$$

$$= \int_{t-1}^{t} x(s)\,ds \qquad f(t) =$$

$$= f(s)\Big|_{t-1}^{t} = f(t) - f(t-1)$$

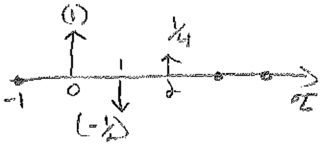
HW# 3.6

$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$

$h(t) = \delta(t) - \frac{1}{2}\delta(t-1) + \frac{1}{4}\delta(t-2)$

try "fixing" h

$h(\tau) =$



$y(0) = 1$

$y(1) = -\frac{1}{2}$

$y(2) = \frac{1}{4}$

$$y(t) = \begin{cases} 0 & t<0 \\ 1 & 0 \le t < 1 \\ -1/2 & 1 \le t < 2 \\ 1/4 & 2 \le t < 3 \\ 0 & t < 3 \end{cases}$$

10/2 - Lecture

$\boxed{\text{causality}}$

$$x \rightarrow \boxed{H} \rightarrow y$$

causal if current and past outputs do not depend on future inputs

ie No peeking ahead!

Causality for LTI system

$x(n) \rightarrow \boxed{H} \rightarrow y(n) = \sum h(k)x(n-k)$

$\therefore y(n) = \cdots + \underbrace{h(-2)x(n+2) + h(-1)x(n+1)}_{\text{future values of } x} + h(0)x(n) + \cdots$ $\therefore$ NOT causal

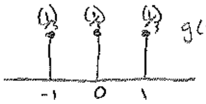
---

☆ A DT LTI system H is causal iff its impulse response is $h(n) = 0 \quad \forall n < 0$ ☆

Same applies to CT

☆ A CT LTI system G is causal iff its impulse response is $g(t) = 0 \quad \forall t < 0$

$$y(n) = \frac{\overset{\text{current}}{x(n)} + \overset{\text{past}}{x(n-1)}}{2} \text{ is } \underline{\text{causal}}$$

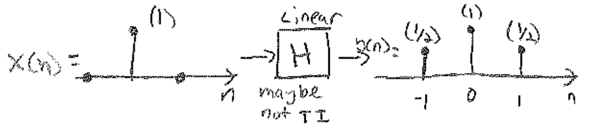


$$y(n) = \frac{\overset{\text{future}}{x(n+1)} + \overset{\text{current}}{x(n)} + \overset{\text{past}}{x(n-1)}}{3} \rightarrow \text{not } \underline{\text{causal}}$$

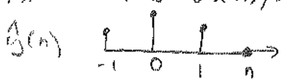
☆ if output does not depend on $x(t)/x(n) \rightarrow$ Causal ☆

IIR: $y(n) = \alpha y(n-1) + x(n) \quad y(n) = 0 \ \forall n < 0$

$h(n) = \alpha^n u(n) \rightarrow \underline{\text{causal}}$



consider $\hat{x}(n) = 2x(n)$, output must be $\hat{y}(n) = 2y(n)$



up to $n = -1$, $x(n) = \hat{x}(n)$, but $\hat{y}(-1) \neq y(-1)$

Not causal!

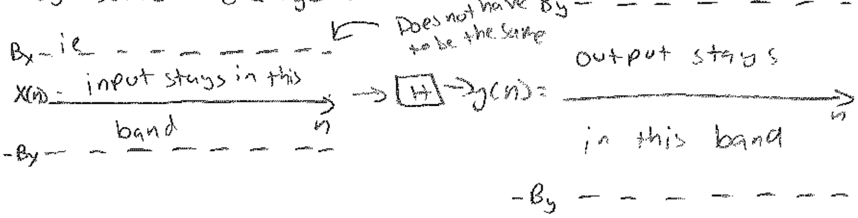
if $x(n) = 0 \ \forall n$ is applied to a linear system, $y(n) = 0 \ \forall n$, (ZIZO): Zero Input → Zero output

---

Bounded Input → Bounded Output (BIBO) stability

X is bounded if $\exists \ 0 < B_x < \infty \quad |x(n)| \leq B_x \quad \forall n \in \mathbb{Z}$

H is BIBO stable if $\forall x \in X$ s.t. X is bounded by a $0 < \underset{\text{positive finite value}}{B_x} < \infty$, then the corresponding y is bounded by some $0 < B_y < \infty$



$y(n) = nx(n) \ \forall n \in \mathbb{Z} \quad x(n) = 1 \ \forall n, \ y(n) = n \ \forall n$, NOT Bounded

NOT BIBO stable.

$y(n) = e^{x(n)} \rightarrow$ let $|x(n)| \leq B_y \Rightarrow y(n) \leq e^{B_x} = B_y$, $\underline{\text{YES BIBO}}$

BIBO for DT LTI: H is BIBO stable iff the impulse response is absolutely summable (ie converges)

If $\sum_n |h(n)| < \infty \Rightarrow$ H is BIBO stable $\forall$ BI → BO

$y(n) = \sum_k h(k)x(n-k) \qquad |x(n)| \leq B_x < \infty$

$|y(n)| = \left|\sum_k h(k)x(n-k)\right| \leq \sum_k |h(k)x(n-k)| \leq \sum_k |h(k)||x(n-k)|$

☆ triangle inequality ☆

$\therefore \sum_k |h(k)| B_x \Rightarrow |y(n)| \leq B_x \sum_k |h(k)| \Rightarrow$ finite $\quad \sum_k |h(k)| B_x$

Show DT LTI BIBO stable $\Rightarrow \sum_n |h(n)| < \infty$

$\neg \sum_n |h(n)| < \infty \Rightarrow H \neg$ BIBO stable.

$\exists$ a bounded input X that produces an unbounded y.

Let $x(n) \begin{cases} \frac{h(-n)}{|h(-n)|} & \text{if } h(-n) \neq 0 \\ 0 & \text{if } h(-n) = 0 \end{cases}$ x(n) can only be 0 if $h(-n) = 0$, 1 if $h(n) > 0$, -1, if $h(-n) < 0$

Clearly, $|x(n)| \leq 1 \Rightarrow$ bounded.

$y(\ ) = \sum_k h(k)x(-k) = \sum_k h(k)\,\mathrm{sgn}(h(k)) = \sum_k |h(k)| \not< \infty$

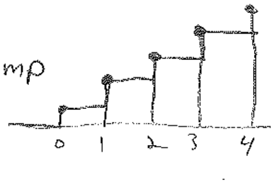
$y(n) = \alpha y(n-1) + x(n) \qquad y(n) = 0 \quad \forall n < 0$

$h(n) = \alpha^n u(n) \qquad \sum_{n=\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |\alpha|^n = \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha < 1 \\ \infty & \text{if } \alpha \geq 1 \end{cases}$

if $\alpha = 1$, $h(n) = u(n) \Rightarrow \sum_n |h(n)| \Rightarrow \infty$

let $x(n) = 1 \ \forall n \Rightarrow y(n) = \sum_{k=\infty}^{\infty} u(k) \rightarrow \infty$

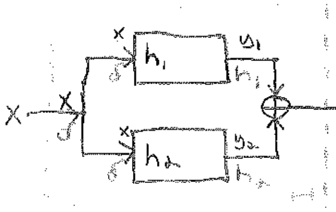
$x(n) = u(n)$; $y(n) =$ ramp



10/7 – Lecture

Interconnections of LTI systems
- Parallel
- cascade (series)
- Feed back (we focus on this)

Parallel



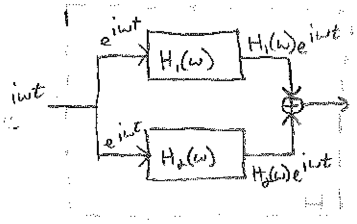
$y = y_1 + y_2$

$h = h_1 + h_2$

H has an impulse response: $h = h_1 + h_2$

what about $H(\omega)$?

$H(\omega) = H_1(\omega) + H_2(\omega)$



$y = H_1(\omega)e^{i\omega t} + H_2(\omega)e^{i\omega t}$

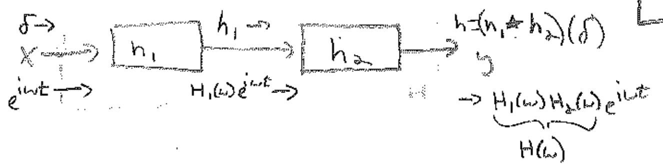
$= e^{i\omega t}\underbrace{(H_1(\omega)+H_2(\omega))}$

$= e^{i\omega t} H(\omega)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty}[h_1(t) + h_2(t)]e^{i\omega t}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} h_1(t)\, e^{-i\omega t}\, dt}_{H_1(\omega)} + \underbrace{\int_{-\infty}^{\infty} h_2(t)\, e^{-i\omega t}\, dt}_{H_2(\omega)}$$

Cascade



$h(t) = (h_1 \star h_2)(t) \xleftrightarrow{\mathcal{F}} H(\omega) = H_1(\omega) H_2(\omega)$

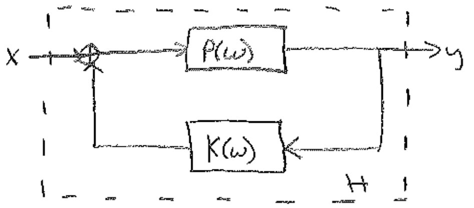
★ convolution in time corresponds to multiplication in frequency ★

Q: Does the order of h_1/h_2 in the cascade matter?

Ans: NO! $h_1 \star h_2 = h_2 \star h_1$

$H_1(\omega)H_2(\omega) = H_2(\omega)H_1(\omega)$

Feedback



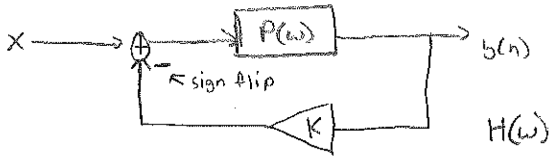
Determine $H(\omega)$

$e^{i\omega t} \rightarrow \quad P(\omega)e^{i\omega t} + K(\omega)e^{i\omega t}P(\omega) = H(\omega)e^{i\omega t}$

$y(t) = P(\omega)[1 + H(\omega)K(\omega)]\cancel{e^{i\omega t}} = H(\omega)\cancel{e^{i\omega t}}$

$P(\omega) = H(\omega) - P(\omega)H(\omega)K(\omega)$

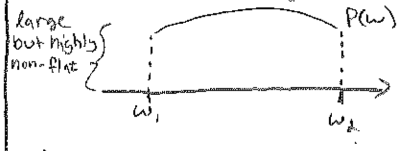
★ $H(\omega) = \frac{P(\omega)}{1 - P(\omega)K(\omega)} = \frac{\text{Foward gain}}{1 - \text{loop gain}}$ $\boxed{\text{Blacks Formula}}$ ★



$H(\omega) = \frac{P(\omega)}{1 + K\,P(\omega)}$

↑ sign flip.

what can you say about $H(\omega)$ if $|kP(\omega)| \gg 1$?

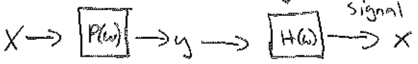


$H(\omega) \approx \frac{1}{K}$

★ for amplification, all we need is for K to be less than 1 ★

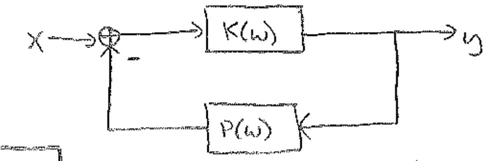
Compensation for non Ideal elements

↓ ie need to amplify signal to account for signal strength loss.



ie want $H(\omega) \approx \frac{1}{P(\omega)}$

?



$y = \frac{K(\omega)}{1 + K(\omega)P(\omega)}$

if $|K(\omega)P(\omega)| \gg 1$ ↗ over the frequency range of interest.

then $H(\omega) \approx \frac{1}{P(\omega)}$

and signal is amplified.

10/8 - Discussion

this example is feeding an "impulse train" like function in.

$$\sum_{k=0}^{K} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

initially at rest, causal, BIBO stable.

assume $\underline{a_0 = 1}$

a) Let $K=3$, and $M=2$

$H(\omega) = ?$

$\overset{1}{\cancel{a_0}} y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 y(n-3)$

$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$

sub $x(n) = e^{i\omega n}$ ↗

$y(n) = H(\omega) e^{i\omega n}$

$(\therefore)$ $H(\omega)\cancel{e^{i\omega n}} + a_1 H(\omega)\cancel{e^{i\omega(n-1)}} + a_2 H(\omega)\cancel{e^{i\omega(n-2)}} + a_3 H(\omega)\cancel{e^{i\omega(n-3)}}$

$= b_0 \cancel{e^{i\omega n}} + b_1 \cancel{e^{i\omega(n-1)}} + b_2 \cancel{e^{i\omega(n-2)}}$

$H(\omega) + a_1 H(\omega) e^{-i\omega} + a_2 H(\omega) e^{-i\omega 2} + a_3 H(\omega) e^{-i\omega 3}$

$= b_0 + b_1 e^{-i\omega} + b_2 e^{-i\omega 2}$

$(\therefore)$ $H(\omega)\left(1 + a_1 e^{-i\omega} + a_2 e^{-2i\omega} + a_3 e^{-i3\omega}\right) = b_0 + b_1 e^{-i\omega} + b_2 e^{-i2\omega}$

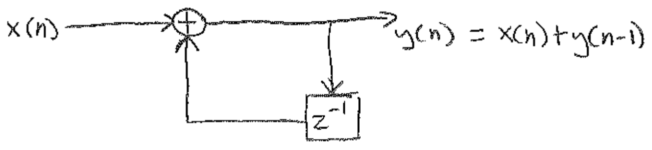
$$\boxed{H(\omega) = \frac{b_0 + b_1 e^{-i\omega} + b_2 e^{-2i\omega}}{1 + a_1 e^{-i\omega} + a_2 e^{-2i\omega} + a_3 e^{-3i\omega}}}$$

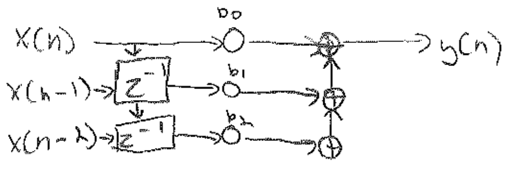
$$(\therefore)\ H(\omega) = \frac{\sum_{m=0}^{M} b_m e^{-i\omega m}}{\sum_{k=0}^{K} a_k e^{-ik\omega}}$$

Blacks Formula Recap:

$$\boxed{H(\omega) = \frac{\text{foward gain}}{1 - \text{loop gain}}}$$

Delay adder gain Diagram





10/9 - Lecture

$$x(n) \longrightarrow \overset{LTI}{\boxed{H(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sum_m x(m)\delta(n-m) \to y(n) = \sum_m x(m)h(n-m)$$

$$x(n) = \alpha_0 e^{i\omega_0 n} + \alpha_1 e^{i\omega_1 n} \longrightarrow \boxed{H(\omega)} \to y(n)?$$

$$y(n) = \cancel{H(\omega)}\, x(n)$$ ☆ <u>NOT TRUE IN GENERAL</u>☆

True if $x(n) = \alpha e^{i\omega n}$

ie $X(n)$ consists of <u>ONLY ONE</u> frequency

ie $X(n) = \alpha_0 e^{i\omega_0 n} \to \boxed{H(\omega)} \to y(n) = \alpha_0 H(\omega_0) e^{i\omega_0 n}$

<u>So</u>: $x(n) = \alpha_0 e^{i\omega_0 n} + \alpha_1 e^{i\omega_1 n} \longrightarrow \boxed{H(\omega)} \to y(n) = ?$

☆ if the filter doesn't like ω_0 then the value will be low or zero ☆

$$y(n) = \alpha_0 H(\omega_0) e^{i\omega_0 n} + \alpha_1 H(\omega_1) e^{i\omega_1 n}$$

what if:

$$x(n) = \sum_k X_k e^{i\omega_k n} \longrightarrow \boxed{H} \to y(n) = \sum_k \underbrace{X_k H(\omega_k)}_{Y_K} e^{i\omega_k n}$$

$$y(n) = \sum_k Y_k e^{i\omega_k n}$$

ex: Suppose $P=3$

what are the freq_s present in X?

Fundamental frequency $= \frac{2\pi}{P} = \frac{2\pi}{3}$

$\therefore$ $0, \frac{2\pi}{3}, \frac{4\pi}{3}$

what about $\frac{6\pi}{3}$? $= 2\pi$ $\quad \therefore e^{i(\omega+2\pi)n} = e^{i\omega n} \cancelto{1}{e^{i2\pi w}}$

So we can also say that in the 3-periodic signal X above, can have at most the freq_s: $\frac{-2\pi}{3}, 0, \frac{2\pi}{3}$

$-\frac{2\pi}{3}$? $\Rightarrow e^{i(\frac{4\pi}{3} - 2\pi)n} = e^{i(-\frac{2\pi}{3})n}$

$e^{i\omega(n+P)} = e^{i\omega n} e^{i\omega P n}$ $\quad \omega P = 2\pi \quad \therefore e^{i\omega n} \cancelto{1}{e^{i\omega P n}}$

$\left(\begin{matrix}\text{Assume } \omega \text{ is a} \\ \text{rational multiple} \\ \text{of } \pi\end{matrix}\right) \Rightarrow e^{i\omega n}$ → periodic w/ period 2π in ω
→ periodic w/ period ϕ in n

Can we index k over a contiguous set of p integers aside from $\overbrace{(0,1,2,\cdots,P-1)}^{\langle P\rangle}$?

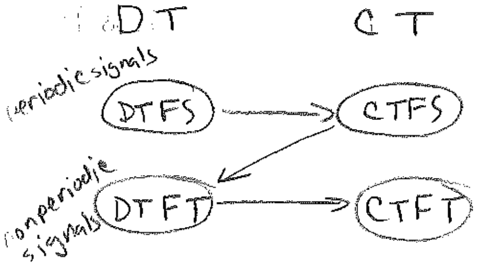
Ans: YES → $\overbrace{(1,2,3,\cdots,P)}^{\langle P\rangle}$, $\underbrace{(2,3,4,\cdots P, P+1)}_{\langle P\rangle} \cdots$

$e^{i\underline{P}\omega_0 n} = e^{i2\pi n} = 1 = e^{i\underline{0}\omega_0 n}$

notation: $\langle P\rangle$: is a contiguous set of P integers.

$\therefore X(n) = \sum_{k=\langle P\rangle} X_k e^{ik\omega_0 n}$

Fourier Analysis Road Map



A DT Periodic Signal:

$[x(n+P) = x(n)$, for some positive int P, $\forall n \in \mathbb{Z}]$

Can be decomposed as $\underline{x(n) = \sum_{k=0}^{P-1} X_k e^{ik\omega_0 n}}$

$P\omega_0 = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{P}$
↑ fundamental frequency

☆ we choose p to be the fundamental Period ☆

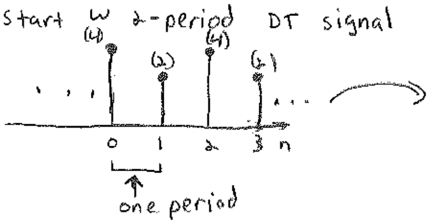
The <u>only</u> frequencies present in x are the <u>fundamental frequencies</u>

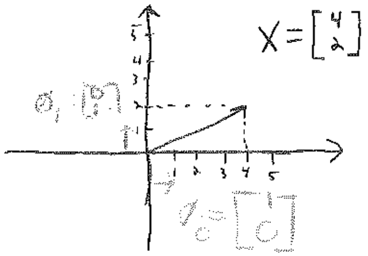
ie: $0\cdot\omega_0, 1\cdot\omega_0, 2\cdot\omega_0, 3\cdot\omega_0, \cdots, (P-1)\omega_0$

Q: what contribution do the signals give?

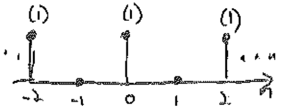
lets think of them as vectors

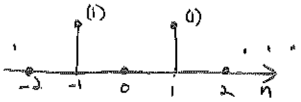
start w 2-period DT signal





$\therefore X = X_0\phi_0 + X_1\phi_1 = 4\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$

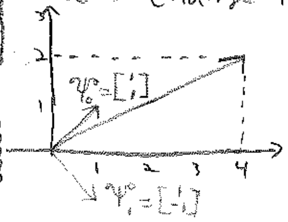
$\phi_0(n) =$


$\phi_1(n) =$


$\therefore X(n) = \overset{X_0}{4}(\phi_0)(n) + \overset{X_1}{2}(\phi_1)(n)$

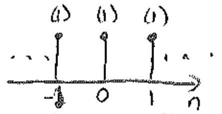
Spliting into impulses
Decomposition of X in terms of shifted impulses

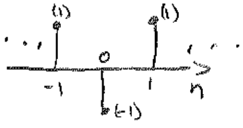
lets change the Question



$\hat{X} = \begin{bmatrix}4\\2\end{bmatrix}$ $\quad \begin{array}{cc} a & b \\ \end{array}$ $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$

$\hat{X} = X_0\hat{\psi}_0 + X_1\hat{\psi}_1$

$\psi_0(n) = e^{i0n}$


$\psi_1(n) = e^{i\pi n} =$


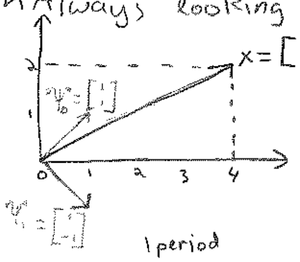
2 periodic signals are expandable in terms of $\omega_0 = \frac{2\pi}{2} = \pi$

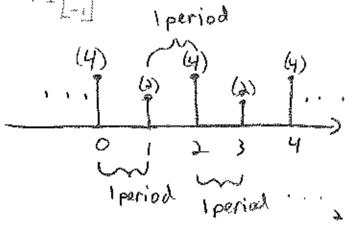
$$x(n) = X_0\psi_0(n) + X_1\psi_1(n) = X_0 e^{i0n} + X_1 e^{i\pi n}$$

10/14 – Lecture

Vector Analysis continued.

☆ Always looking at the fundamental P ☆



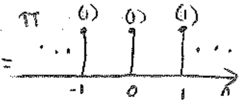


Can be represented as a $\overset{\text{2 values before repeat}}{2}$D cartesian

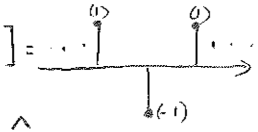
ie: $X(n+2) = x(n) \quad \forall n \in \mathbb{Z}$

$p = 2, \quad \omega_0 = \frac{2\pi}{2} = \pi$

$e^{i0n} \rightarrow \psi_0(n) = \begin{bmatrix}1\\1\end{bmatrix} =$



$e^{i\pi n} \rightarrow \psi_1(n) = \begin{bmatrix}1\\-1\end{bmatrix} =$



$\hat{X} = X_0 \hat{\psi}_0 + X_1 \hat{\psi}_1$

$\begin{bmatrix}4\\2\end{bmatrix} = X_0 \begin{bmatrix}1\\1\end{bmatrix} + X_1 \begin{bmatrix}1\\-1\end{bmatrix}$

$x(n) = X_0 \psi_0(n) + X_1 \psi_1(n) \quad \forall n \in \mathbb{Z}$

Goal: Find X_0, X_1

$x(n) = X_0 e^{i0n} + X_1 e^{i\pi n}$

x is decomposable into the frequency components: 0 and π

☆ In general, a p-periodic signal can have at most the following frequencies:
$0, \ \omega_0 = \frac{2\pi}{P}, \ 2\omega_0, \ldots, (P-1)\omega_0$ ☆

☆ no other frequency is possible! ☆

$(\therefore)\ x(n) = X_0 e^{i0n} + X_1 e^{i\omega_0 n} + X_2 e^{i2\omega_0 n} + \ldots + X_{p-1} e^{i(P-1)\omega_0 n}$
$= \sum_{k=0}^{P-1} X_k e^{ik\omega_0 n}$

Back to the 2 periodic example: ☆geometrically project☆

$x = \begin{bmatrix}4\\2\end{bmatrix} = X_0 \begin{bmatrix}1\\1\end{bmatrix} + X_1 \begin{bmatrix}1\\-1\end{bmatrix}$

to determine X_0, project $\hat{x}$ onto ψ_0

$(\therefore)\ \hat{x} \cdot \hat{\psi}_0 = \begin{bmatrix}4\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = (4)(1) + (2)(1) = 6 = LHS$

$= \underbrace{\left(X_0 \begin{bmatrix}1\\1\end{bmatrix} + X_1 \begin{bmatrix}1\\-1\end{bmatrix}\right)}_{x} \cdot \begin{bmatrix}1\\1\end{bmatrix} \quad RHS$

☆ Dot Product of orthogonal = ∅ ☆

$= X_0 [1\ 1]\begin{bmatrix}1\\1\end{bmatrix} + X_1 [1\ -1]\begin{bmatrix}1\\1\end{bmatrix} = 2X_0$

$(\therefore)\ 2X_0 = 6 \rightarrow X_0 = 3$

To Determine X_1, project $\hat{x}$ onto $\hat{\psi}_1$

$\hat{x} \cdot \psi_1 = \begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = (4)(1) + (2)(-1) = 2$

$= \left(X_0 \begin{bmatrix}1\\1\end{bmatrix} + X_1 \begin{bmatrix}1\\-1\end{bmatrix}\right) \cdot \begin{bmatrix}1\\-1\end{bmatrix}$

$= X_0 [1\ 1]\begin{bmatrix}1\\-1\end{bmatrix} + X_1 [1\ -1]\begin{bmatrix}1\\-1\end{bmatrix} = 2X_1$

$(\therefore)\ 2X_1 = 2 \rightarrow X_1 = 1$

$(\therefore)\ x(n) = 3\,\psi_0(n) + 1\,\psi_1(n)$

$= \underset{X_0}{3} e^{i0n} + \underset{X_1}{1} e^{i\pi n}$

w/ Period = 2

$\hat{\psi}_0(n) = e^{i0n}$

$\hat{\psi}_0 = \begin{bmatrix}\psi_0(0)\\\psi_0(1)\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$\hat{\psi}_1 = \begin{bmatrix}\psi_1(0)\\\psi_1(1)\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

---

what if $p = 3$? $\rightarrow \omega_0 = \frac{2\pi}{3} \rightarrow \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$

$\psi_0(n) = e^{i\emptyset n}$

$\psi_1(n) = e^{i\frac{2\pi n}{3}}$

$\psi_2(n) = e^{i\frac{4\pi n}{3}}$

$\hat{X} = \begin{bmatrix}x(0)\\x(1)\\x(3)\end{bmatrix}$

$\hat{\psi}_1 = \begin{bmatrix}\psi_1(0)\\\psi_1(1)\\\psi_1(2)\end{bmatrix} = \begin{bmatrix}1\\e^{i\frac{2\pi}{3}}\\e^{i\frac{4\pi}{3}}\end{bmatrix}$

$\hat{\psi}_0 = \begin{bmatrix}\psi_0(0)\\\psi_0(1)\\\psi_0(2)\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$\hat{\psi}_2 = \begin{bmatrix}\psi_2(0)\\\psi_2(1)\\\psi_0(2)\end{bmatrix} = \begin{bmatrix}1\\e^{i\frac{4\pi}{3}}\\e^{i\frac{2\pi}{3}}\end{bmatrix}$

$X(n) = X_0 \psi_0(n) + X_1 \psi_1(n) + X_2 \psi_2(n) = X_0 e^{i0n} + X_1 e^{i\frac{2\pi n}{3}} + X_2 e^{i\frac{4\pi n}{3}}$

Assume for now that $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$ are $\overbrace{\text{mutually orthogonal}}^{\text{prove this later}}$

Q: How would you determine X_0, X_1, X_2 in $\hat{X} = X_0 \hat{\psi}_0 + X_1 \hat{\psi}_1 + X_2 \hat{\psi}_2$

Ans: PROJECT x onto ψ_k to determine X_k.

Dot product can no longer be used! why?

$a \cdot a = \|a\|^2$ with equality iff $a = 0$

we must generalize the notion of the dot product.

Inner Product: $\langle \hat{a}, \hat{b} \rangle \triangleq \hat{a}^T \cdot \hat{b}^*$

$\hat{a} = \begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}, \hat{b} = \begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix}, \langle \hat{a}, \hat{b} \rangle = [a_0\ a_1\ a_2]\begin{bmatrix}b_0^*\\b_1^*\\b_2^*\end{bmatrix} = \sum_{k=0}^{n} a_k b_k^*$

Assume $\psi_k \perp \psi_\ell \rightarrow k \neq \ell$ (ie $\langle \hat{\psi}_k, \hat{\psi}_\ell \rangle = 0$)

To determine X_k, project $\hat{X}$ onto $\hat{\psi}_k$:

$\langle \hat{x}, \hat{\psi}_k \rangle = \langle X_0\psi_0 + X_1\psi_1 + X_2\psi_2, \psi_k \rangle$

$= X_k \langle \psi_k, \psi_k \rangle$ ☆ all other terms are zero ☆ b/c all vectors are orthogonal

$k = 0, 1, 2, \ldots, P-1$

$(\therefore)\ X_k = \frac{\langle \hat{x}, \hat{\psi}_k \rangle}{\langle \hat{\psi}_k, \hat{\psi}_k \rangle}$

$\psi_k = \begin{bmatrix}1\\e^{ik\omega_0}\\\vdots\\e^{ik\omega_0(P-1)}\end{bmatrix}$

$(\therefore)\ \hat{X} = \begin{bmatrix}x(0)\\x(1)\\\vdots\\x(P-1)\end{bmatrix} \quad \psi_0 = \begin{bmatrix}\psi_0(0)\\\psi_0(1)\\\vdots\\\psi_0(P-1)\end{bmatrix} \ldots \psi_k = \begin{bmatrix}\psi_k(0)\\\psi_k(1)\\\vdots\\\psi_k(P-1)\end{bmatrix} \ldots \psi_{P-1} = \begin{bmatrix}\psi_{P-1}(0)\\\psi_{P-1}(1)\\\vdots\\\psi_{P-1}(P-1)\end{bmatrix}$

what is $\langle \hat{\psi}_k, \hat{\psi}_k \rangle$?: $[1\ e^{ik\omega_0} \ldots e^{ik\omega_0(P-1)}]\begin{bmatrix}e^{-ik\omega_0}\\\vdots\\e^{-ik\omega_0(P-1)}\end{bmatrix}$ $= \sum_{k=0}^{P-1} X_k \psi_k(n) = \boxed{\sum_{k=0}^{P-1} X_k e^{ik\omega_0 n}}$

Synthesis equation

$\langle \hat{x}, \hat{\psi}_k \rangle = \hat{x}^T \hat{\psi}_k^* = [x(0) \cdots x(P-1)]\begin{bmatrix}1\\e^{-ik\omega_0}\\\vdots\\e^{-ik\omega_0(P-1)}\end{bmatrix} = \sum_{n=0}^{P-1} x(n) e^{-ik\omega_0 n} \Rightarrow X_k = \frac{\langle \hat{x}, \hat{\psi}_k \rangle}{\langle \hat{\psi}_k, \hat{\psi}_k \rangle} = \boxed{\frac{1}{P} \sum_{n=0}^{P-1} x(n) e^{-ik\omega_0 n}}$

Analysis equation

10/16 - Lecture

DTFS Review

Synthesis: $x(n) = \sum_{k=0}^{p-1} \overset{\text{what we are looking for}}{X_k} \underbrace{e^{ik\omega_0 n}}_{\psi_k(n)}$

Analysis: $X_k = \frac{1}{p} \sum_{n=0}^{p-1} x(n) e^{-ik\omega_0 n}$

$$\hat{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(p-1) \end{bmatrix}, \quad \psi_k = \begin{bmatrix} \psi_k(0) \\ \psi_k(1) \\ \vdots \\ \psi_k(p-1) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{ik\omega_0} \\ \vdots \\ e^{ik\omega_0(p-1)} \end{bmatrix}$$

$(\therefore)\ \psi_k(n) = e^{ik\omega_0 n}$

$$\hat{x} = \sum_{k=0}^{p-1} X_k \hat{\psi}_k = X_0 \hat{\psi}_0 + X_1 \hat{\psi}_1 + \cdots + X_{p-1} \hat{\psi}_{p-1}$$

Inner Product

$$\langle \hat{f}, \hat{g} \rangle = \hat{f}^T g^* = \begin{bmatrix} f(0) & f(1) & \cdots & f(p-1) \end{bmatrix} \begin{bmatrix} g^*(0) \\ g^*(1) \\ \vdots \\ g^*(p-1) \end{bmatrix}$$
$$= \sum_{n=0}^{p-1} f(n) g^*(n)$$

Can define inner products for signals (and functions) in the same way

if f, g are periodic signals in p, $\langle \hat{f}, \hat{g} \rangle \triangleq \sum_{n=0}^{p-1} f(n) g^*(n)$

$(\therefore)$ signal $x = \sum_{k=0}^{p-1} X_k \psi_k$

Last time: assumed $\psi_k \perp \psi_l \quad k \neq l$

$\langle \psi_k, \psi_k \rangle = \|\psi_k\|^2 = p \qquad \langle \psi, \psi_l \rangle_{k=l} = 0$

General Compact form:

$\langle \psi_k, \psi_l \rangle = p\delta(k-l)$

lets show mutual orthogonality:

$$\langle \psi_k, \psi_l \rangle = \sum_{n=0}^{p-1} \psi_k(n) \psi_l^*(n) = \sum_{n=0}^{p-1} e^{ik\omega_0 n} e^{-il\omega_0 n}$$
$$= \sum_{n=0}^{p-1} e^{i(k-l)\omega_0 n} = \sum_{n=0}^{p-1} 1 = p \quad \text{if } k=l$$

Detour.........

$$S = \sum_{k=A}^{B} \alpha^k = \alpha^A + \alpha^{A+1} + \cdots + \alpha^B$$

if $\alpha = 1 \Rightarrow S = B - A + 1$

Assume $\alpha \neq 1$

$$\alpha S = \cancel{\alpha^{A+1}} + \cancel{\alpha^{A+2}} + \cdots + \cancel{\alpha^{B}} + \alpha^{B+1}$$
$$-\ S = \alpha^A + \cancel{\alpha^{A+1}} + \cancel{\alpha^{A+2}} + \cdots \cancel{\alpha^B} \quad \leftarrow \text{subtract}$$

$(\alpha - 1)S = \alpha^{B+1} - \alpha^A$

$$S = \frac{\alpha^{B+1} - \alpha^A}{\alpha - 1}$$

$$(\therefore)\ \frac{\left(e^{i(k-l)\omega_0}\right)^p - 1}{e^{i(k-l)\omega_0} - 1} = \frac{\cancel{e^{i(k-l)\overset{2\pi}{\omega_0 p}}}^{\,1} - 1}{\text{dont care/nonzero}} = 0 \quad (\therefore)\ \psi_k \perp \psi_l \quad k \neq l \quad k, l = 0, 1, 2, \cdots, p-1$$

$x = X_0 \psi_0 + \cdots + X_{p-1} \psi_{p-1} \qquad \langle \psi_k, \psi_l \rangle = p\delta(k-l)$

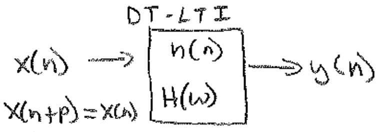
To determine X_k, project each side onto ψ_k:

$$\langle x, \psi_k \rangle = \langle X_0 \psi_0 + \cdots + X_{p-1} \psi_{p-1}, \psi_k \rangle$$
$$= X_0 \langle \psi_0, \psi_k \rangle + \cdots + X_{p-1} \langle \psi_{p-1}, \psi_k \rangle$$
$$= X_k \langle \psi_k, \psi_k \rangle$$

☆ All other terms zero b/c all values of $\langle \psi_l, \psi_k \rangle$ w/ $l \neq k = \emptyset$

$$(\therefore)\ X_k = \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} = \frac{1}{p} \langle x, \psi_k \rangle$$
$$= \frac{1}{p} \sum_{n=0}^{p-1} x(n) \psi_k^* = \frac{1}{p} \sum_{n=0}^{p-1} x(n) e^{-ik\omega_0 n}$$

Periodic Signals Through LTI systems.



☆ must be periodic Also ☆

$x(n+p) = x(n)$

$\forall n \in \mathbb{Z} \quad \exists p > 0 \quad p$: integer

$$x(n) = \sum_k X_k e^{ik\omega_0 n} \rightarrow y(n) = \sum_k \underbrace{X_k H(k\omega_0)}_{Y_k} e^{ik\omega_0 n}$$

$y(n) = \sum_k Y_k e^{ik\omega_0 n}$

$y(n+p) = y(n)$

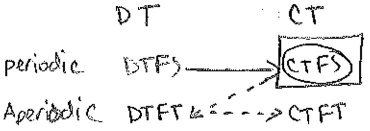
$Y_k = X_k H(k\omega_0)$

☆ Alternative way ☆

$$\hat{x}(n) = x(n+p) \rightarrow \boxed{H, h} \rightarrow \hat{y}(n) \overset{\uparrow}{=} y(n+p)$$

by T-I

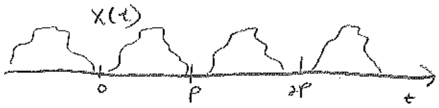
$= x(n)$ ↑ we know → A system is a function → $\hat{y}(n) = y(n)$



$\omega_0 = \frac{2\pi}{p}$

for Ct, can also write

$p = \frac{2\pi}{\omega_0}$



$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{\infty} X_k \psi_k(t)$$

Discrete: $x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} = \sum_{k=0}^{p-1} X_k \psi_k(n)$

Our space: set of periodic CT signals $\qquad f, g$: p-periodic CT signals

$\langle f, g \rangle \triangleq \int_0^p f(t) g^*(t)\, dt$ Ⓒ(CT)

$\langle f, g \rangle = \sum_{n=0}^{p-1} f(n) g^*(n)$ (DT)

$\langle f, f \rangle = \|f\|^2 \int_0^p f(t) f^*(t)\, dt = \int_0^p |f(t)|^2\, dt$ ☆ energy of f in one period ☆

10/21 - Lecture

$\boxed{\text{CTFS}}$

$x(t+p) = x(t) \quad \forall t \in \mathbb{R}, \; p > 0$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \underbrace{e^{ik\omega_0 t}}_{\psi_k \text{ functions}} \qquad \omega_0 = \frac{2\pi}{p}$$

$\psi_k(t) = e^{ik\omega_0 t}$

$\boxed{\text{GOAL: Determine } X_k}$

$x(t) = \sum X_k \psi_k(t) = \sum_k X_k \psi_k$

Assume for now : $\psi_k \perp \psi_\ell \quad k \neq \ell$

in DT: $\langle f, g \rangle = \sum_{n=0}^{p-1} f(n) g^*(n)$

$\|f\|^2 = \langle f, f \rangle = \sum_{n=0}^{p-1} |f(n)|^2$

in CT: (p-periodic CT signal space)

$\langle f, g \rangle \triangleq \int_0^p f(t) g^*(t) \, dt$

$\langle f, f \rangle \triangleq \int_0^p |f(t)|^2 dt \leftarrow$ "energy of the signal"

Back to $x = \sum_k X_k \psi_k$

to determine X_k, project x onto ψ_ℓ

$\langle x, \psi_\ell \rangle = \langle \sum_k X_k \psi_k , \psi_\ell \rangle = \sum_k X_k \langle \psi_k, \psi_\ell \rangle$

$= \sum_k X_k \langle \psi_\ell, \psi_\ell \rangle$ b/c only one value is orthogonal

$(\therefore) \; X_\ell = \frac{\langle x, \psi_\ell \rangle}{\langle \psi_\ell, \psi_\ell \rangle} \rightsquigarrow X_k = \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle}$

$$\langle \psi_k, \psi_k \rangle = \int_{\langle p \rangle} \overbrace{\psi_k(t) \psi_k^*(t)}^{\text{all have same magnitude}} dt = \int_{\langle p \rangle} |\psi_k(t)|^2 \, dt = \int_{\langle p \rangle} 1 \, dt = \boxed{p}$$

$\langle p \rangle$: is a continous interval of duration p

$X_k = \frac{1}{p} \langle x, \psi_k \rangle = \frac{1}{p} \int_{\langle p \rangle} x(t) \cdot \psi_k^*(t) \, dt$

$(\therefore) \; X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt$

$\boxed{(\therefore) \text{ CTFS Equations}}$

$\boxed{\text{Synthesis:}}$ $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$

$\boxed{\text{Analysis:}}$ $X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt$

☆Only the following set of frequencies can be present: $\ldots, -2\omega_0, -\omega_0, 0, \omega_0, \ldots$ ☆

show that $\psi_k \perp \psi_\ell$ when $k \neq \ell$

$$\langle \psi_k, \psi_\ell \rangle = \int_{\langle p \rangle} \psi_k(t) \psi_\ell^*(t) \, dt = \int_{\langle p \rangle} e^{i(k-\ell)\omega_0 t} \, dt = \int_0^p e^{i(k-\ell)\omega_0 t} dt$$

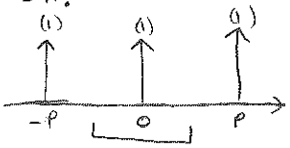
$$= \left. \frac{e^{i(k-\ell)\omega_0 t}}{i(k-\ell)\omega_0} \right|_0^p = \frac{\overset{\rightarrow 1}{e^{i(k-\ell)\overset{2\pi}{\omega_0 p}}} - 1}{i(k-\ell)\omega_0} = 0$$

$\int_0^p \cos[(k-\ell)\omega_0 t] \, dt + i \int_0^p \sin[(k-\ell)\omega_0 t] \, dt$

integrate over an integer number of complete cycles of the sine and cosine.

$\langle \psi_k, \psi_\ell \rangle = p \delta(k - \ell) \qquad \psi_k(t) = e^{ik\omega_0 t}$

Ex:



$= \quad x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t - \ell p)$

$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$

$X_k = ?$

pick an appropriate interval of p ie, $-\frac{p}{2} \rightarrow \frac{p}{2}$

then I'm dealing with the one impulse

$$X_k = \frac{1}{p} \int_{-p/2}^{p/2} \underset{\uparrow}{\delta(t)} e^{-ik\omega_0 t} \, dt = \frac{1}{p}$$

only valid @ $t = 0$, by sifting

$(\therefore) \; x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t - \ell p) = \frac{1}{p} \sum_{k=-\infty}^{\infty} e^{ik\omega_0 t}$

10/22 — Discussion

Recall the CTFS:

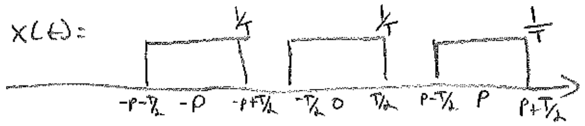
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

$$X_k = \frac{1}{P}\int_{\langle P\rangle} x(t)\, e^{-ik\omega_0 t}\, dt$$

Find the CTFS and X_k

A): $x(t) = \sin(\frac{2\pi t}{3})$

B) $x(t)$:



A) $x(t) = \sin(\frac{2\pi t}{3})$ $\qquad \frac{2\pi}{P} = \frac{2\pi}{3}^{\omega_0}$ $\therefore P=3$

$$X_k = \frac{1}{P}\int_0^P \sin(\frac{2\pi t}{3}) e^{ik\omega_0 t}\, dt$$

$$= \frac{1}{2P_i}\int_0^P \left(e^{i\frac{2\pi t}{3}} - e^{-i\frac{2\pi t}{3}}\right)\left(e^{-ik\omega_0 t}\right) dt$$

$$= \frac{1}{2i}\left[e^{i\frac{2\pi t}{3}} - e^{-i\frac{2\pi t}{3}}\right]$$

$\therefore X_1 = \frac{1}{2i}$, $X_0 = 0$, $X_{-1} = \frac{-1}{2i}$, $X_k = 0 \ \forall k \neq 1,-1$

b/c signal is already a set of complex exponentials $\therefore$ you can find X_k.

method 1:

$\text{sinc}(x) = \frac{\sin(x)}{x}$ $\qquad \omega_0 = \frac{\omega T}{2}$

$$X(\omega) = \text{sinc}(\frac{\omega T}{2}) = \frac{2}{\omega T}\sin(\frac{\omega T}{2})$$

show that $X_k = \frac{1}{P} X(\omega)\big|_{\omega = k\omega_0}$

$$\frac{2}{2i\omega T}\left[e^{i\frac{\omega T}{2}} - e^{-i\frac{\omega T}{2}}\right]$$

$\therefore X_1 = \frac{1}{i\omega T}$, $X_{-1} = \frac{1}{i\omega T}$

$X_k = 0 \ \forall k \neq 1,-1$

B) method 2

$$X_k = \frac{1}{P}\int_{-P/2}^{P/2} x(t)\, e^{-ik\omega_0 t}\, dt$$

$$= \frac{1}{PT}\int_{-T/2}^{T/2} \cancel{\tfrac{1}{T}}\, e^{-ik\omega_0 t}\, dt = \left(\frac{1}{TPik\omega_0}\right)(-1)\left(e^{-ik\omega_0 t}\right)\Big|_{-T/2}^{T/2}$$

$$= \frac{-1}{TPik\omega_0}\left[e^{-i\frac{k\omega_0 T}{2}} - e^{i\frac{k}{2}\omega_0 T}\right]$$

$$= \frac{2}{k\omega_0 PT}\sin\left(\frac{k\omega_0 T}{2}\right) = \frac{1}{P}\frac{2}{k\omega_0 T}\sin\left(\frac{k\omega_0 T}{2}\right) = \frac{1}{P}\text{sinc}\left(\frac{k\omega_0 T}{2}\right)$$

$$\lim_{T\to 0}\frac{1}{P}\frac{2}{k\omega_0 T}\sin\left(\frac{k\omega_0 T}{2}\right) = \frac{1}{P}\lim_{T\to 0}\frac{\cancel{\frac{k\omega_0}{2}}\cos\left(\frac{k\omega_0 T}{2}\right)}{\cancel{\frac{k\omega_0}{2}}} \quad \frac{1}{P}X(k\omega_0)$$

$$= \left(\frac{1}{P}\right)(1) = \boxed{\frac{1}{P}}$$

Linearity:

$x, y \Longleftrightarrow X_k, Y_k$

$$\alpha_1 x_1 + \alpha_2 y \xleftrightarrow{\mathcal{F}} \alpha_1 X_k + \alpha_2 Y_k$$

$$r(t) = \alpha_1 x(t) + \alpha_2 y(t)$$

$$R_k = \underbrace{\frac{\alpha_1}{P}\int_{\langle P\rangle} x(t) e^{-ik\omega_0 t}\, dt}_{\alpha_1 X_k} + \underbrace{\frac{\alpha_2}{P}\int_{\langle P\rangle} y(t) e^{-ik\omega_0 t}\, dt}_{\alpha_2 Y_k}$$

Time shifting:

$$Y(t) = x(t-t_0) \xleftrightarrow{\mathcal{F}} Y_k = X_k e^{-ik\omega_0 t_0}$$

$$Y_k = \frac{1}{P}\int_{\langle P\rangle} x(t-t_0) e^{-ik\omega_0 t}\, dt \qquad \text{let } u = t - t_0$$

$t = u + t_0$

$du = dt.$

Time reversal: $y(t) = x(-t) \xleftrightarrow{\mathcal{F}} x - k?$

$$Y_k = \frac{1}{P}\int_{\langle P\rangle} x(-t) e^{-ik\omega_0 t}\, dt \qquad \text{let } u = -t$$

$-du = dt$

$$= \frac{1}{P}\int_{\langle P\rangle} x(u) e^{-i(-k)\omega_0 t}\, dt$$

$$= X_{-k}$$

Time scaling: $y(t) = x(\alpha t) = \sum_{k=-\infty}^{\infty} \underbrace{X_k}_{\text{doesn't change}} e^{ik\underbrace{(\omega_0\alpha)}t}$ frequency changes.

fyi → not LTI.

multiplication:

$$r(t) = x(t)y(t) \xleftrightarrow{\mathcal{F}} R_x = \sum_{m=-\infty}^{\infty} X_m Y_{k-m}$$

$$r(t) = x(t)\,y(t) = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} X_k Y_l e^{i(k+l)\omega_0 n} \qquad \text{let } l = m-k$$

$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} X_k Y_{m-k} e^{i\omega_0 mn}$$

10/23 - Lecture

CTFS:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

$$X_k = \frac{1}{P}\int_{<P>} x(t)\, e^{-ik\omega_0 t}\, dt$$

$$\Psi_k(t) = e^{ik\omega_0 t}$$

$$\Psi_k(t+P) = e^{ik\omega_0 t}\, \cancel{e^{ik\overbrace{\omega_0 P}^{2\pi}}} \to 1 = \Psi_k(t) \quad \forall t$$

$\Psi_k(t)$ not periodic in $\boxed{k}$ ✯

$$\Psi_{k+p}(t) = e^{i(k+p)\omega_0 t} = e^{ik\omega_0 t}\, \underbrace{e^{ip\overbrace{\omega_0}^{2\pi} t}}_{\neq 1 \text{ if } \underline{t} \text{ is not integer!}}$$

$\Psi_k(t)$ not periodic in ω_0

proof by contridiction: assume $\exists\ \lambda > 0$

s.t. $e^{ik(\omega_0 + \lambda t)} = e^{ik\omega_0 t} \quad \forall t$

∴ no such λ exists

$\boxed{\text{Discrete}}$:

$$\Psi_k(n) = e^{ik\omega_0 n}$$

$$e^{ik(\omega_0 + 2\pi)n} = e^{ik\omega_0 n}\, \cancel{e^{ik2\pi n}}^{\,1}$$

$\Psi_k(n)$ is 2π periodic in ω_0

$$\Psi_k(n+P) = e^{ik\omega_0 n}\, \cancel{e^{ik\underbrace{\omega_0 P}_{2\pi}}} \to 1$$

$\Psi_k(n)$ is periodic in n

k and n serve mathematically identical roles in the exponent

$$\Psi_{k+P}(n) = \Psi_k(n)$$

$\Psi_k(n)$ is periodic in $\underline{k}$

Convergence of the CTFS

$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$ approx to x

↑ $\qquad e_n(t) = x(t) - x_n(t)$ error

should not be interpreted in the usual point-wise sense.

✯If x has finite energy in one period

$$\int_{<P>} |x(t)|^2\, dt < \infty$$

then $\lim_{N\to\infty} \int_{<P>} |e_N(t)|^2\, dt = 0$

the error energy goes to zero as the number of terms, N, goes to infinity.

A second type of Convergence

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

✯ equallity holds for all but a countable (Discrete) set of

✯ points along the time axis. (in one period) due to Dirichlet.

If 3 Dirichlet conditions hold, then the above convergence holds

① $\int_{<P>} |x(t)|\, dt < \infty \qquad x(t) = \begin{cases} 1/t & 0 < t \le 1 \end{cases}$ periodically replicates.

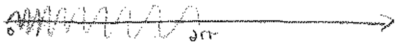
if $\int_{<P>} |x(t)|\, dt < \infty \Rightarrow X_k = \frac{1}{P}\left|\int_{<P>} x(t)\, e^{-ik\omega_0 t}\, dt\right| \le \frac{1}{P}\int_{<P>} \left|x(t)\, e^{-ik\omega_0 t}\right| dt$

$= \frac{1}{P}\int_{<P>} |x(t)| \cdot \left|\cancel{e^{-ik\omega_0 t}}\right| dt$

∴ $|X_k| \le \frac{1}{P}\int_{<P>} |x(t)|\, dt < \infty$

② x must be of bounded variation in one period. x has a finite number of minimum and maxima in one period.

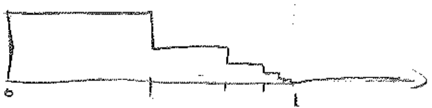
Counter-example: $x(t) = \sin\left(\frac{2\pi}{t}\right)$



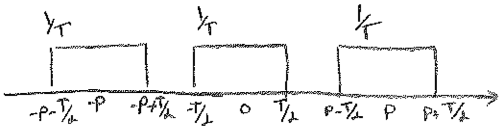
③ In one period, x must have a finite # of discontinuities

counter-example:

x loses half stregnth w/ half time remaining?



Pulse Train Example.



Determine X_k

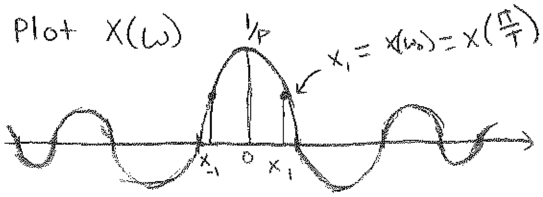
$$X_k = \frac{1}{P}\int_{<P>} x(t)\, e^{-ik\omega_0 t}\, dt \Rightarrow X(\omega) = \frac{1}{P\frac{\omega}{2}T}\sin\left(\frac{\omega T}{2}\right) = \frac{2}{P\omega T}\sin\left(\frac{\omega T}{2}\right)$$

$X_k = X(k\omega_0) = X(\omega)\Big|_{\omega = k\omega_0} \qquad X(0) = ?$

lopitals rule AKA "French Hospital"

$$\lim_{\omega\to 0} \frac{\sin\left(\frac{\omega T}{2}\right)}{P\left(\frac{\omega T}{2}\right)} = \boxed{\frac{1}{P}}$$

Plot $X(\omega)$



$P = 2T \qquad \omega_0 = \frac{2\pi}{P} = \frac{\cancel{2}\pi}{\cancel{2}T} = \frac{\pi}{T}$

$X_2 = 0 = X_{-2} \qquad X\left(\pm\frac{2\pi}{T}\right)$

10/28 - Lecture

Fourier Analysis

	DT	CT
Periodic	DTFS/DFT →	CTFS
not periodic	$\boxed{\text{DTFT}}$ ←	CTFT
	here ↑ now	

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-i\omega k} \rightarrow \text{transform to}$$

we're after an expression for h(n)

ω is a continous variable $(\therefore)$ expression will be an integral!!

want to show: $h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)e^{i\omega n}\, dw$

$$H(\omega) = \sum_k h(k)\underbrace{e^{-i\omega k}}_{\phi_k(\omega)} \qquad (\therefore)\ H(\omega) = \sum_k h(k)\,\phi_k(\omega)$$

if $\phi_k \perp \phi_n$ $n \neq k$

⟨Project!!⟩ to determine h(k)

$$\langle H, \phi_n\rangle = \langle \sum_k h(k)\,\phi_k, \phi_n\rangle$$
$$= \sum_k h(k)\langle \phi_k, \phi_n\rangle$$
$$= h(n)\langle \phi_n, \phi_n\rangle$$

b/c $\phi_k \perp \phi_n$ except for k=n

$$\langle H, \phi_n\rangle = h(n)\langle \phi_n, \phi_n\rangle$$
$$(\therefore)\ h(n) = \frac{\langle H, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$$

✯ we live in the universe of 2π-periodic ★ functions of ω.

$$\phi_k(\omega+2\pi) = e^{-i(\omega+2\pi)k} = e^{-i\omega k}\,\cancel{e^{-i2\pi k}}^{\,1}$$
$$\phi_k(\omega+2\pi) = e^{-i\omega k} = \phi_k(\omega)$$

recall: $\langle F, G\rangle \triangleq \int_{\langle 2\pi\rangle} F(\omega)\, G^*(\omega)$ $\leftarrow p = 2\pi$

$$\langle \phi_n, \phi_n\rangle = \int_{\langle 2\pi\rangle} \phi_n(\omega)\,\phi_n^*(\omega)\, dw = \int_{\langle 2\pi\rangle} e^{-i\omega n} e^{i\omega n}\, dw = \boxed{2\pi}$$

$$h(n) = \frac{1}{2\pi}\langle H, \phi_n\rangle = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)\,\phi_n^*(\omega)\, dw$$

$$\boxed{h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)e^{i\omega n}\, dw \leftarrow \text{Synthesis}} \ ★$$

$$\boxed{H(\omega) = \sum_{k=-\infty}^{\infty} h(n)\, e^{-i\omega n} \leftarrow \text{Analysis}} \ ★$$

For a signal x: $X(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\, e^{i\omega n}\, dw$

$\boxed{\text{Spectrum of X}} \rightarrow X(\omega) = \sum_n x(n)\, e^{-i\omega n}$

X(ω) is the DTFT of X(n)

Interpretation of the Synthesis:

$$x(n) = \int_{\langle 2\pi\rangle} \underbrace{\frac{dw}{2\pi}\ X(\omega)}\, e^{i\omega n}$$

↑ Linear combinations of complex exponentials!!

DTFS

$$X(n) = \sum_{k\in\langle p\rangle} X_k\, e^{ik\omega_0 n}$$

CTFS

$$X(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{ik\omega_0 t}$$

$\frac{dw}{2\pi}$ is common ∀ω.

what distinguishes one freq ω_1 from another, ω_2?

$$X(\omega_1) \text{ vs } X(\omega_2)$$

Back to showing $\phi_k \perp \phi_n$ $(n \neq k)$

$$\langle \phi_k, \phi_n\rangle = \int_{\langle 2\pi\rangle} \phi_k(\omega)\,\phi_n^*(\omega)\, dw = \int_{\langle 2\pi\rangle} e^{-i\omega k} e^{i\omega n}\, dw = \int_{\langle 2\pi\rangle} e^{i(n-k)\omega}\, dw$$

From the CTFS to the DTFT:

$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{ik\omega_0 t}$ where x is periodic

$X_k = \frac{1}{p}\int_{\langle p\rangle} X(t)e^{-ik\omega_0 t}\, dt$ suppose: $p = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{p} = 1$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ikt} \qquad X_k = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} x(t)\, e^{-itk}\, dt$$

let $t = \omega$

$$(\therefore)\ x(\omega) = \sum_{k=-\infty}^{\infty} X_k\, e^{i\omega k} \qquad X_k = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} x(\omega)\, e^{-i\omega k}\, dw$$

let $x = X$

$$X(\omega) = \sum_{k=-\infty}^{\infty} X_k\, e^{i\omega k} \qquad X_k = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{-i\omega k}\, dw$$

let $k = -n$

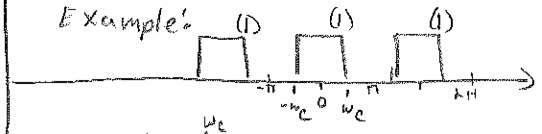
$$X(\omega) = \sum_{n=-\infty}^{\infty} X_{-n}\, e^{-i\omega n} \qquad X_{-n} = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\, e^{i\omega n}\, dw$$

let $x(n) = X_{-n}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-i\omega n} \qquad X(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\, e^{i\omega n}\, dw$$

$(\therefore)$ $\overset{\to X(\omega)}{\text{DTFT}}$ is nothing new!

Example:



H(ω) Ideal low Pass filter.

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega n}\, dw$$

$$= \frac{1}{2\pi}\left(\frac{e^{i\omega n}}{in}\Big|_{-\omega_c}^{\omega_c}\right) = \frac{e^{i\omega_c n} - e^{-i\omega_c n}}{2i\,\pi n} = \frac{\sin(\omega_c n)}{2\pi}$$

$$h(n) = \frac{\sin(\omega_c n)}{2\pi} \overset{\mathcal{F}}{\longleftrightarrow} ?$$

$\sum_n |h(n)|$ doesn't converge → filter is not BIBO stable

IIR filter cannot be implimented w/a differencing equation.

10/29 - Discussion

Shifting / scaling property

$$\mathcal{F}\{x(n-M)\} = X(\omega)e^{-i\omega M}$$

★ shift in time is a scaling in frequency★

Ex: $\mathrm{Re}\{X(\omega)\} = \frac{1}{2} - \frac{1}{2}\cos(\omega)$

x(n) is real, causal, find x(n)

hint: $\frac{X(\omega) + X^*(\omega)}{2} = \mathrm{Re}\{X(\omega)\}$

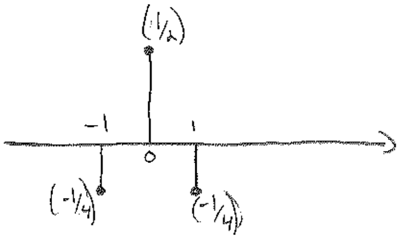
hint: $X(\omega) = X^*(-\omega)$

hint: $\mathcal{F}^{-1}\{X(\omega)\}$ use DTFS

$$\frac{X(\omega) + X(-\omega)}{2} = \frac{1}{2} - \frac{e^{i\omega} + e^{-i\omega}}{4}$$

$$\frac{X(\omega) + X(-\omega)}{2} = \frac{1}{2} - \frac{e^{i\omega} + e^{-i\omega}}{4} \xleftrightarrow{\mathcal{F}} \frac{x(n) + x(-n)}{2} = \frac{\delta(n)}{2} - \frac{\delta(n+1) + \delta(n-1)}{4}$$

$$\frac{x(n) + x(-n)}{2} =$$



since x(n) is causal, $x(n) = 0$ for $n < 0$

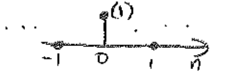
$$x(n) = \frac{\delta(n)}{2} - \frac{\delta(n-1)}{2}$$

10/30 - Lecture

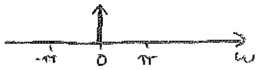
DTFT:

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega n}\,d\omega$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

Ex: $x(n) = \delta(n)$

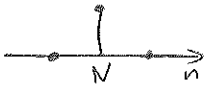


$$\rightarrow X(\omega) = \sum \delta(n)e^{-i\omega n}$$
$$= \delta(0)e^{-i\omega 0} = 1$$



narrow in frequency ⟺ wide in time

Ex: $\hat{x}(n) = x(n-N) = \delta(n-N)$



$$\xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = \sum_n \delta(n-N)e^{-i\omega n}$$
$$= e^{-i\omega N}$$

$$x(n) \xleftrightarrow{\mathcal{F}} X(\omega)$$
$$\hat{x}(n) = x(n-N) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = \sum_n x(n-N)e^{-i\omega n}$$

Let $l = n-N$ $= \sum_{l=-\infty}^{\infty} x(l)e^{-i\omega(l+N)}$

$$= \underbrace{\boxed{\sum_l x(l)e^{-i\omega l}}}_{X(\omega)} e^{-i\omega n}$$

$$(\therefore)\ \hat{x}(n) = x(n-N) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega)e^{-i\omega n}$$

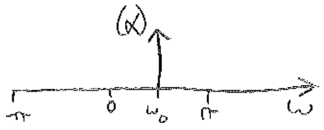
$$\hat{x}(n) = \underline{\quad ? \quad} \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega-\omega_0)$$

$$\hat{x}(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\lambda)e^{i(\lambda+\omega_0)n}\,d\lambda$$
$$= \underbrace{\boxed{\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\lambda)e^{i\lambda n}\,d\lambda}}_{x(n)} e^{i\omega_0 n}$$

$$(\therefore)\ \hat{x}(n) = x(n)e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega-\omega_0)$$

frequency shift property

Ex: $x(n) = e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}} X(\omega) = ?$



$X(\omega) = \delta(\omega-\omega_0)$ $|\omega| < \pi$

periodically replicating.

Ex continued: Try analysis:

$$X(\omega) = \sum_{n=-\infty}^{\infty} e^{i\omega_0 n}e^{-i\omega n} = \sum_{n=-\infty}^{\infty} e^{i(\omega_0-\omega)n} \rightarrow \text{doesnt converge}$$

Try: Synthesis:

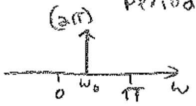
$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} \underbrace{\alpha\delta(\omega-\omega_0)}_{X(\omega)} e^{i\omega_0 n}\,d\omega$$
$$= \frac{\cancel{\alpha}^{\,k2\pi}}{2\pi} e^{i\omega_0 n} \overset{\checkmark}{=} x(n) = e^{i\omega_0 n}$$

$(\therefore)$ $\alpha = 2\pi$

$$x(n) = e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}} X(\omega) = \begin{cases} 2\pi\delta(\omega-\omega_0) & |\omega| < \pi \\ 2\pi\text{-periodic replicating} \end{cases}$$

$$X(\omega) = 2\pi\sum_{l=-\infty}^{\infty} \delta(\omega-\omega_0 \underbrace{-2\pi l}_{\text{periodic}})$$

$e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}}$

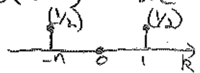


DTFT of Periodic signals:

$$x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n}$$
$$e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0)$$
$$e^{ik\omega_0 n} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-k\omega_0)$$
$$X_k e^{ik\omega_0 n} \xleftrightarrow{\mathcal{F}} 2\pi\cdot X_k\cdot\delta(\omega-k\omega_0)$$
$$x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n} \xleftrightarrow{\mathcal{F}} X(\omega) = 2\pi\sum_k X_k\delta(\omega-\omega_0)$$
$$X(\omega) = \sum_n x(n)e^{-i\omega n}$$
$$2X(\omega) = \sum_n 2x(n)e^{-i\omega n} \longrightarrow \text{general} \rightarrow$$
$$x(n) = x_1(n) + x_2(n)$$
$$X(\omega) = X_1(\omega) + X_2(\omega)$$

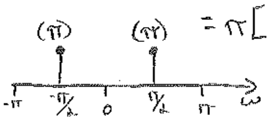
Ex: $\cos(\frac{\pi}{2}n) = \frac{1}{2}e^{i\frac{\pi}{2}n} + \frac{1}{2}e^{-i\frac{\pi}{2}n}$ $(\therefore)$ $\omega_0 = \frac{-\pi}{2}, 0, \frac{\pi}{2}$ $p=4$

$(\therefore)$ $X_k =$

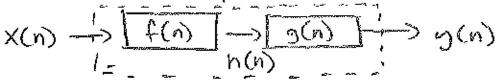


$X_{-1} = \frac{1}{2}$, $X_0 = 0$, $X_1 = \frac{1}{2}$

$$X(\omega) = 2\pi X_{-1}\delta(\omega+\tfrac{\pi}{2}) + 2\pi X_1\delta(\omega-\tfrac{\pi}{2})$$
$$= \pi[\delta(\omega+\tfrac{\pi}{2}) + \delta(\omega-\tfrac{\pi}{2})]$$



Convolution prop of the DTFT



$h(n) = (f*g)(n)$ $\quad h(n) = (f*g)(n) \xleftrightarrow{\mathcal{F}} H(\omega) = F(\omega)G(\omega)$

$H(\omega) = F(\omega)G(\omega)$ ☆ convol in time ⟷ multi in frequency ☆

Recall: $x(n-N) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-i\omega n}$

$y(n) = (x*h)(n) \xleftrightarrow{\mathcal{F}} Y(\omega) = X(\omega)H(\omega)$

Application to LCCDE s:

$y(n) = \alpha y(n-1) + x(n)$ $\quad |\alpha| < 1$ $\quad y(n) = 0$ $\quad n < 0$

Take the DTFT of both sides and see if they are the same.

$Y(\omega) = \alpha Y(\omega)e^{-i\omega} + X(\omega)$ $\qquad$ Recall: $Y(\omega) = X(\omega)H(\omega)$

$X(\omega)H(\omega) = \alpha Y(\omega)e^{-i\omega} + X(\omega)$

$$H(\omega) = \alpha \underbrace{\boxed{\frac{Y(\omega)}{X(\omega)}}}_{H(\omega)\Rightarrow} e^{-i\omega} + 1 \rightarrow \boxed{H(\omega) = \frac{1}{1-\alpha e^{-i\omega}}}$$

11/4 - lecture

Some Convergence Issues with DTFT

$X(\omega) = \sum_{n=-\infty}^{\infty} X(n) e^{-i\omega n}$

Tamest signal is absolutely sumable.

$\left(\sum_{n=-\infty}^{\infty} |X(n)|\right) < \infty$ ← This is also BIBO stable.

ℓ_1 = set of all signals from $\mathbb{Z}$ to $\mathbb{C}$

s.t. it is absolutely sumable.

ie. $\ell_1 = \{x: \mathbb{Z} \to \mathbb{C} \mid \sum_n |x(n)| < \infty\}$

ℓ_1 signals have DTFTs that are:

ⓐ finite: if x is ℓ_1 $(x \in \ell_1)$ then $|X(\omega)| < \infty \ \forall \omega$

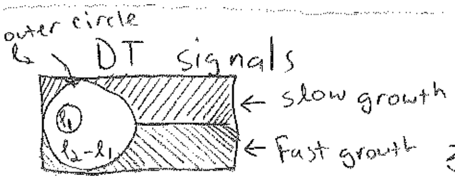
ⓑ x is continous in ω

Proof for ⓐ $X(\omega) = \sum_n x(n) e^{-i\omega n}$

$|X(\omega)| = \left|\sum_n x(n) e^{-i\omega n}\right| \leq \sum_n |X(n) e^{-i\omega n}|$

$= \sum_n |X(n)| \cdot \cancel{|e^{-i\omega n}|}^{1}$

$(\therefore)$ if x is ℓ_1 then $|X(\omega)| < \infty$



$\underbrace{\text{Z transform}}$

eg: $h(n) = \alpha^n u(n) \quad |\alpha| > 1$

Examples of ℓ_1 signals:

$X(n)\ \delta(n) \longrightarrow X(\omega) = 1$

$g(n) = \alpha^n u(n) \longrightarrow G(\omega) = \frac{1}{1 - \alpha e^{-i\omega}} \quad |\alpha| < 1$

ℓ_2 signals: $\left(\sum_{n=-\infty}^{\infty} |X(n)|^2\right) < \infty$ $\boxed{\text{Finite energy signals}}$

DTFT must be carefully defined:

★ cant use the analysis equation ★

can use synthesis withot worry

use a partial sum: let $X_N(\omega) = \sum_{n=-N}^{N} X(n) e^{-i\omega n}$

then $X(\omega) = \lim_{n \to \infty} X_n(\omega)$ → converge in some sense.

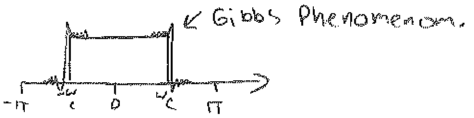
★ if x is ℓ_1 as well ★ convergence is uniform

peak difference between X_n & X goes to $\emptyset$ $\forall \omega$

$\lim_{N \to \infty} |X(\omega) - X_n(\omega)| = 0$

★ if x is ℓ_2 but not ℓ_1

Ex: $h(n) = \frac{\sin(\omega_c n)}{\pi n}$ ideal LPF $H(\omega)$



★ no Periodic signal can be ℓ_1 ★
b/c not Absolutely summable.

Ex continued: Define Partial sums:

$H_N(\omega) = \sum_{n=-N}^{N} h(n) e^{-i\omega n}$

$\lim_{n \to \infty} \int_{-\pi}^{\pi} |H(\omega) - H_m(\omega)|^2 d\omega = 0$ ↙same as saying

$\lim_{n \to \infty} \sum_n |h(n) - h_N(n)|^2 = 0$

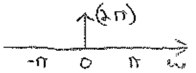
$(\therefore)\ h_N(n) = \begin{cases} \frac{\sin(\omega_c n)}{\pi n} & n \leq N \\ 0 & \text{everywhere else} \end{cases}$

Signals that are neither ℓ_1 nor ℓ_2

- Signals that dont grow faster than polynomial in time (signals of slow growth)

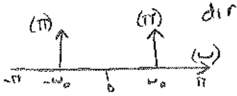
Ex: $x(n) = 1 \ \forall n$ → cant use analysis

$X(\omega) =$

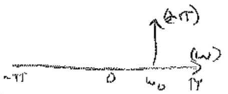


DTFT: can be defined, but it involves dirac deltas.

$X(n) = \cos(\omega_0 n) \longrightarrow$



$g(n) = e^{i\omega_0 n} \longrightarrow$



$y(n) = (x * h)(n) \xrightarrow{\mathcal{F}} Y(\omega) = X(\omega) H(\omega)$ ⇃ Dual of this ⇃

$y(n) = x(n) h(n) \xrightarrow{\mathcal{F}} Y(\omega) = ?$

$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} x(n) h(n) e^{-i\omega n}$

but $x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) e^{i\lambda n} d\lambda$

$(\therefore)\ Y(\omega) = \frac{1}{2\pi} \sum_n h(n) \int_{\langle 2\pi \rangle} X(\lambda) e^{i\lambda n} e^{-i\omega n} d\lambda$

$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) \underbrace{\left[\sum_n h(n) e^{-i(\omega - \lambda)n}\right]}_{H(\omega - \lambda)} d\lambda$$

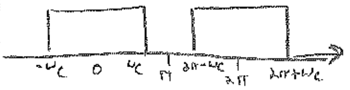
$(\therefore)\ Y(\omega) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\lambda) H(\omega - \lambda) d\lambda = (X \circledast H)(\omega)$ ↙ circular convolution

Recipe:

- Keep $X(\lambda)$ in tack.
- keep only 1 period of $H(\lambda)$
- flip that 1 period of $H(-\lambda)$
- Slide that 1 period of H
- Point-wise multiply: $X(\lambda) H(\omega - \lambda)$
- Integrate as before.

$h(n) = \frac{\sin^2(\omega_c n)}{(\pi n)^2} = g(n) \cdot g(n) \qquad g(n) = \frac{\sin(\omega_c n)}{\pi n}$

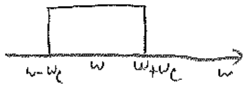
$h(n) =$



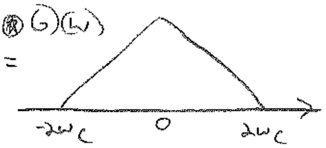
$g(n) =$

$H(\lambda)$ truncated.

$H(\omega - \lambda) =$



⟶ $H(\omega) = (G \circledast G)(\omega)$
=



11/5 – Discussion

$$\cos(\omega_0 n) \rightarrow \underset{\substack{\uparrow \\ \text{real impulse response}}}{\boxed{H}} \rightarrow y(n) = ?$$

$$\downarrow$$

$$\frac{1}{2}e^{i\omega_0 n} + \frac{1}{2}e^{-i\omega_0 n} \rightarrow \boxed{H} \rightarrow \frac{1}{2}e^{i\omega_0 n}H(\omega_0) + \frac{1}{2}e^{-i\omega_0 n}H(-\omega_0)$$

$$\begin{aligned} y(n) &= \frac{1}{2}e^{i\omega_0 n}\;H(\omega_0) + \frac{1}{2}e^{-i\omega_0 n}\;H(-\omega_0) \\ &= \frac{1}{2}e^{i\omega_0 n}H(\omega_0) + \frac{1}{2}e^{-i\omega_0 n}H^*(\omega_0) \\ &= \frac{1}{2}\Big[\underbrace{|H(\omega_0)|}\,e^{i\omega_0 n} + \underbrace{|H^*(\omega_0)|}\,e^{-i\omega_0 n}\Big] \end{aligned}$$

same magnitude

(:) $|H(\omega_0)| \cos(w_0 n + \measuredangle \ldots)$

☆ Use pattern matching when you can ☆

11/6/- Lecture

Parseval's Thm:

finite energy DT signals

$$\langle x, x\rangle = \sum_{n=-\infty}^{\infty} x(n)\,x^*(n) = \sum_n |x(n)|^2 < \infty$$

Show: $\langle x, x\rangle = \frac{1}{2\pi}\langle X, X\rangle$

for discrete aperiodic signals

2π periodic functions of a continous variable

Parsevals Thm:	$\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} \lvert X(w)\rvert^2\, dw$

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(w)\,e^{iwn}\,dw$$

$$x^*(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X^*(w)\,e^{-iwn}\,dw$$

$$(\therefore)\ \sum_n x(n)\,x^*(n) = \frac{1}{2\pi}\sum_n x(n)\int_{\langle 2\pi\rangle} X^*(w)\,e^{-iwn}\,dw$$

$$\sum_n |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X^*(w)\underbrace{\sum_n x(n)\,e^{-iwn}}_{}\,dw$$

$$(\therefore)\ \sum_n |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(w)|^2\,dw \quad X(w)$$

	DT	CT
Periodic	DTFS/DFT →	CTFS
Aperiodic	DTFT ←→	CTFT

here now :)

DTFT:

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(w)\,e^{iwn}\,dw$$

$$X(w) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-iwn}$$

CTFT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\,e^{iwt}\,dw$$

$$X(w) = \int_{-\infty}^{\infty} x(t)\,e^{-iwt}\,dt$$

$$x(t) = \int_{-\infty}^{\infty} \boxed{\frac{dw}{2\pi}\ X(w)}\,e^{iwt}$$ ← linear set of complex exponentials

Spectum: relates to the coefficients, ie: how much of x is in the frequency.

Ex: $x(t) = \delta(t)$

$$X(w_c) = \int_{-\infty}^{\infty} \delta(t)\,e^{-iwt}\,dt = 1 \rightarrow$$ sifting property.

$$\delta(t) \xleftrightarrow{\mathcal{F}} \boxed{1} \qquad \delta(t) = \frac{①}{2\pi}\int_{-\infty}^{\infty} e^{iwt}\,dw$$

$$\delta(t) = \delta(-t) \Rightarrow \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwt}\,dw$$

Ex: $\hat{x}(t) = \delta(t-T) \xrightarrow{\mathcal{F}} \hat{X}(w) = ?$

$$X(w) = \int_{-\infty}^{\infty} \delta(t-T)\,e^{-iwt}\,dt = e^{-iwT}$$

$$\delta(t-T) \xrightarrow{\mathcal{F}} 1 \cdot e^{-iwT}$$

$$\hat{X}(w) = \int_{-\infty}^{\infty} x(t-T)\,e^{-iwt}\,dw$$
$$= \int_{-\infty}^{\infty} x(\tau)\,e^{-iw(\tau+T)}\,d\tau$$
$$= X(w)\,e^{-iwT}$$

(∴) Time shift Property:

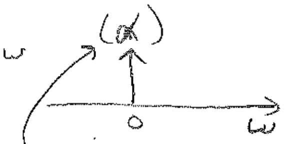
$$x(t) \xleftrightarrow{\mathcal{F}} X(w) \qquad x(t-T) \xleftrightarrow{\mathcal{F}} X(w)\,e^{-iwT}$$

$x(t) = 1 \rightarrow X(w) = ?$

Try synthesis equation.

$$x(t) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\,e^{iwt}\,dw$$

has to be impulse



$$(\therefore)\ 2\pi = \int_{-\infty}^{\infty} X(w)\,e^{iwt}$$

inpulse scalled by something

α

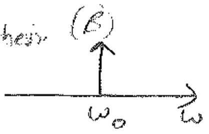
$$2\pi = \int_{-\infty}^{\infty} \alpha\,\delta(w)\,e^{iwt}\,dw \rightarrow 2\pi = \alpha\,\cancel{e^{i0t}}\ 1$$

$(\therefore)\ \alpha = 2\pi$.

$x(t) = 1 \rightarrow 2\pi\,\delta(w)$

Ex: $x(t) = e^{iw_0 t} \xleftrightarrow{\mathcal{F}} X(w) = ?\quad 2\pi\,\delta(w-w_0)$

solve for β using synthesis



☆ multiply in time → shift in frequency ☆

☆ shift in time → multiply in frequency ☆

modulation:

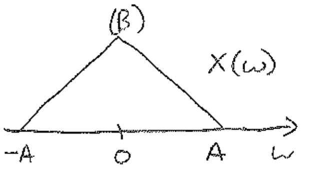
$$x(t) \xleftrightarrow{\mathcal{F}} X(w)$$

$$\hat{x}(t) = x(t)\,e^{iw_0 t} \xleftrightarrow{\mathcal{F}} \hat{X}(w) = X(w-w_0)$$

$$\hat{X}(w) = \int_{-\infty}^{\infty} \underbrace{x(t)\,e^{iw_0 t}}_{\hat{x}(t)}\,e^{-iwt}\,dt$$

$$(\therefore) = \int_{-\infty}^{\infty} x(t)\,e^{-i(w-w_0)t}\,dt$$

$$= X(w-w_0)$$



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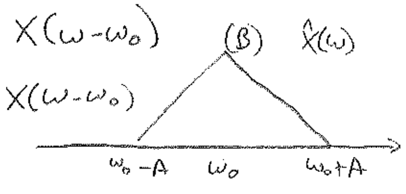
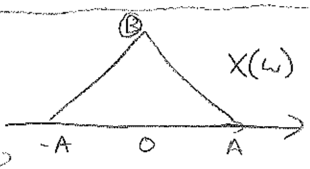
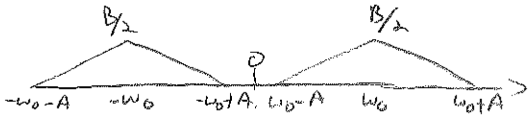


Ex: $x(t) \xleftrightarrow{\mathcal{F}} X(w)$



$$\hat{x}(t) = x(t)\cos(w_0 t) \xleftrightarrow{\mathcal{F}} \hat{X}(w) = ?$$

$$= x(t)\,\tfrac{1}{2}e^{iw_0 t} + \tfrac{1}{2}e^{-iw_0 t} \longrightarrow \tfrac{1}{2}X(w-w_0) + \tfrac{1}{2}X(w+w_0)$$



11/12 - Discussion

(28)

$$\boxed{\hat{x}(t) = x(\alpha t) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)}$$

$$\hat{x}(t) = \frac{dx(t)}{dt}$$

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \frac{d}{dt}\left(e^{i\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, i\omega\, e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega \left(X(\omega) e^{i\omega t} d\omega\right)$$

by pattern matching.

$$\boxed{\hat{x}(t) = \frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = i\omega X(\omega)}$$

$\hat{x}(t) = t\, x(t)$

$\hat{X}(\omega) = i \frac{dX(\omega)}{d\omega}$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt\right] = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega}\left(e^{-i\omega t}\right) dt = \int_{-\infty}^{\infty} -i \underbrace{t\, x(t)}_{\hat{x}(t)} e^{-i\omega t} dt$$

multiply need to get "id" ok

multiply both sides by i

$$(\therefore) \quad i \frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} \cancel{-i^{2}}^{1} \hat{x}(t) e^{-i\omega t} \, dt$$

in general if:

$$\boxed{\begin{aligned} &\hat{x}(t) = t^n x(t) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = i^n \frac{d^n X(\omega)}{d\omega^n} \\ &\hat{x}(t) = x(\alpha t) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right) \\ &\hat{x}(t) = \frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = i\omega X(\omega) \end{aligned}}$$

Orthogonality:

$\phi_k, \ k \in \mathbb{Z}$ $\qquad \psi_k = \int \phi_k e^{i\omega t} dt$

$\phi_k \perp \phi_l \longrightarrow \psi_k \perp \psi_l$

Parcival's Identity

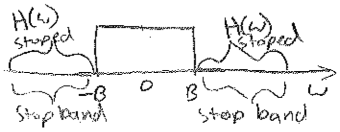
$\langle \phi_k, \phi_l \rangle = \frac{1}{2\pi} \langle \psi_k,$

$\langle e^{ik\omega_0 t}, e^{il\omega_0 t} \rangle$

$$\int_{\langle P \rangle} e^{ik\omega_0 t} e^{-il\omega_0 t} dt = \int_{\langle P \rangle} e^{i(k-l)\omega t} \, dt = \delta(k-l)$$

11/13 - Lecture

Ideal low pass filter

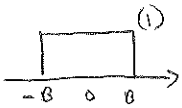


$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-B}^{B} e^{i\omega t}\,d\omega$$

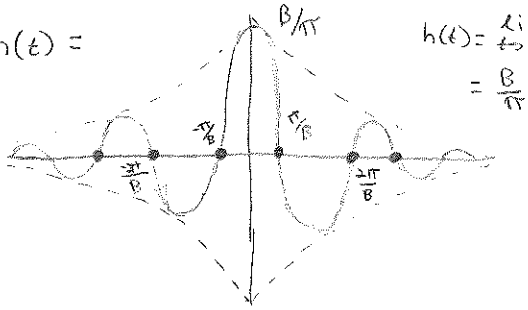
$$= \frac{1}{2\pi}\left.\frac{e^{i\omega t}}{it}\right|_{-B}^{B}$$

$$= \frac{1}{2\pi}\left(\frac{e^{itB}}{it} + \frac{e^{-itB}}{it}\right) = \frac{1}{\pi t}\left(\frac{e^{itB} + e^{-itB}}{2i}\right) = \frac{1}{2\pi}\sin(Bt)$$

$h(t) = \frac{\sin(Bt)}{\pi t} \xleftrightarrow{\mathcal{F}}$



$h(t) =$



$$h(t) = \lim_{t\to 0}\frac{B\cos(Bt)}{\pi} = \frac{B}{\pi}$$

First zero crossing when $Bt = \pi$

if we had $h(t) = \frac{A\sin(Bt)}{\pi t}$ then
$H(\omega)$ is not BIBO stable because
$\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$ is <u>NOT</u> True

H is not Causal b/c $h(t) \neq 0 \ \forall t < 0$
↳ Can't be implimented via LCDDE
∴ H can't be real time filter.

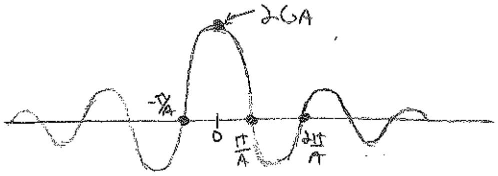
Ex: $g(t) = \begin{cases} G & |t| < A \\ 0 & e/w \end{cases}$ what is $G(\omega)$?

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt = \int_{-A}^{A} Ge^{-i\omega t} = 2G\left(\frac{e^{-i\omega A}}{-2i\omega} - \frac{e^{i\omega A}}{-2i\omega}\right)$$

$$= \frac{2G}{\omega}\left(\frac{e^{i\omega A}}{2i} - \frac{e^{-i\omega A}}{2i}\right) = \frac{2G}{\omega}\sin(\omega A)$$

First zero crossing, when $\omega A = \pi$

$$G(0) = \lim_{\omega\to 0} = \frac{2GA\cancel{\cos(\omega A)}}{1} = 2GA$$



← Back to Ideal Low pass filter

$$\int_{-\infty}^{\infty} h(t)\,dt =$$

$$H(0) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} h(t)\,dt$$

↑ <u>DC gain</u>

$h(0) \to$ DC gain is area under Impulse response

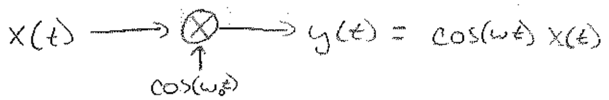
CTFT Properties: Convolution of time

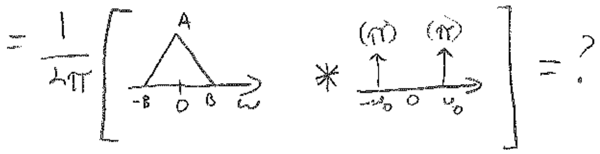
$$h(t) = (f * g)(t) \xleftrightarrow{\mathcal{F}} H(\omega) = F(\omega)G(\omega)$$

$$h(t) = f(t)g(t) \xleftrightarrow{\mathcal{F}} H(\omega) = \frac{1}{2\pi}(X * G)(\omega)$$

$$H(\omega) = \int_{-\infty}^{\infty} x(t)\,g(t)\,e^{-i\omega t}\,dt$$

Ex:





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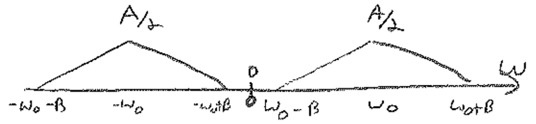
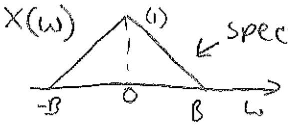


11/18 - Lecture

Amplitude modulation



← spectrum of some voice signal you want to transmit.

$B = 2\pi \cdot 3\times10^3$ rad/s -> 3kHz highest freq.

efficient Antenna ≈ $\frac{1}{4}\lambda$ ← wave length

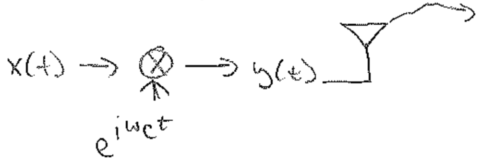
$c = f\lambda$ $c = 3\times10^8$ m

(∴) $\lambda = 100$km

↳ antenna ≈ 25 km long

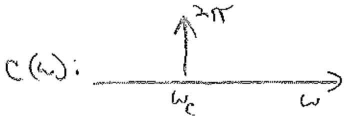
TOO BIG↗

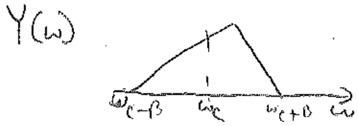
☆that is why we use carrier frequencies☆

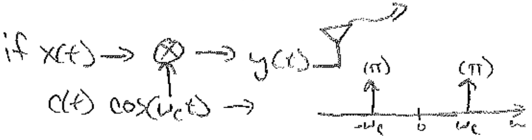


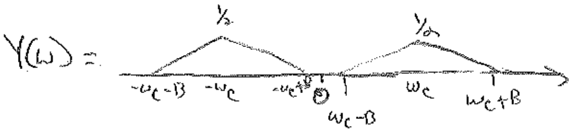
ω_c = carrier frequency

$y(t) = x(t)e^{i\omega_c t} \to Y(\omega) = \frac{1}{2\pi}(X * C)(\omega)$



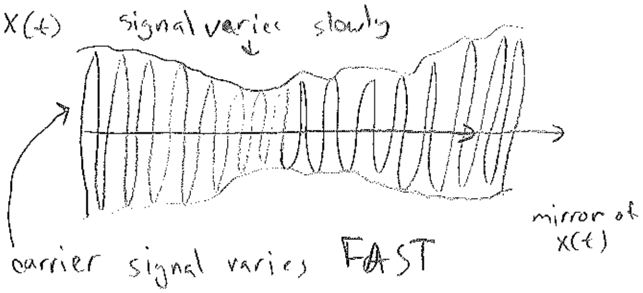






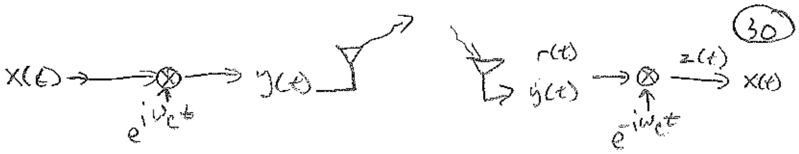
original signal is modulating the amplitude of the carrier signal.

whats the time domain picture?



x→ information bearing signal

c→ carrier (being modulated by x)



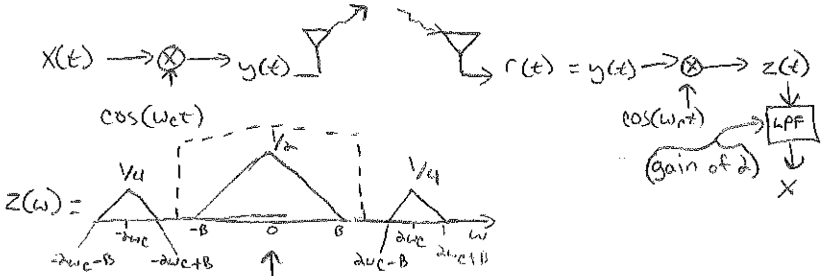
Assumptions:

- y(t) comes in tact
- reciever oscilator
  - oscillates @ exactly ω_c
  - is in phase with transmitter oscilator

$r(t) = y(t)$

$z(t) = r(t)e^{-i\omega_c t} = y(t)e^{-i\omega_c t} = x(t)\cancel{e^{-i\omega_c t}}\,\cancel{e^{i\omega_c t}} = \underline{X(t)}$

Sinusoidal carrier scheme



↑ use a LPF w/gain of 2 to recover this

$\cos^2(\omega_c t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)$

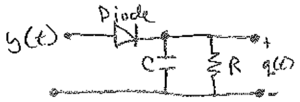
$z(t) = x(t)\cos^2(\omega_c t) = \frac{1}{2}\underset{\text{big triangle}}{x(t)} + \frac{1}{2}x(t)\underset{\text{little triangles.}}{\cos(2\omega_c t)}$

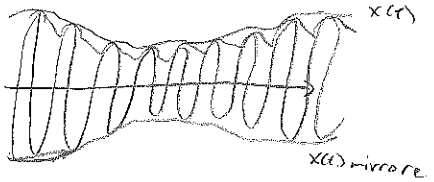
what happens if reciever is out of phase by θ:

$z(t) = x(t)\cos(\omega_c t)\cos(\omega_c t + \theta)$

$= \frac{1}{2}x(t)\underset{\text{high freq term}}{\cos(2\omega_c t + \theta)} + \frac{1}{2}x(t)\underset{\text{hope to catch}}{\cos(\theta)}$ if $\theta = \frac{\pi}{2}$ then scheme fails!

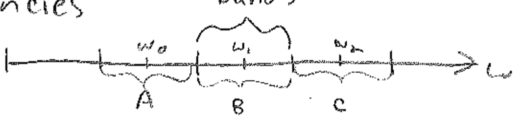
Rectifier scheme:

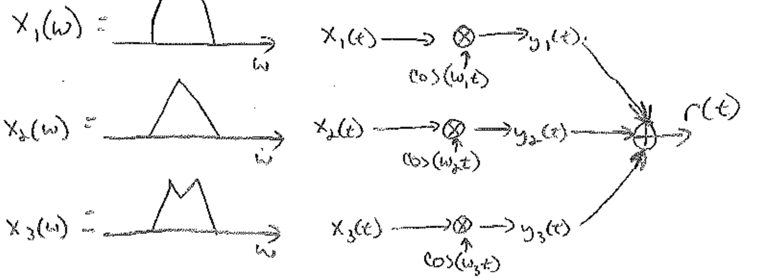




q(t) follows x(t)

"Selling" Frequencies

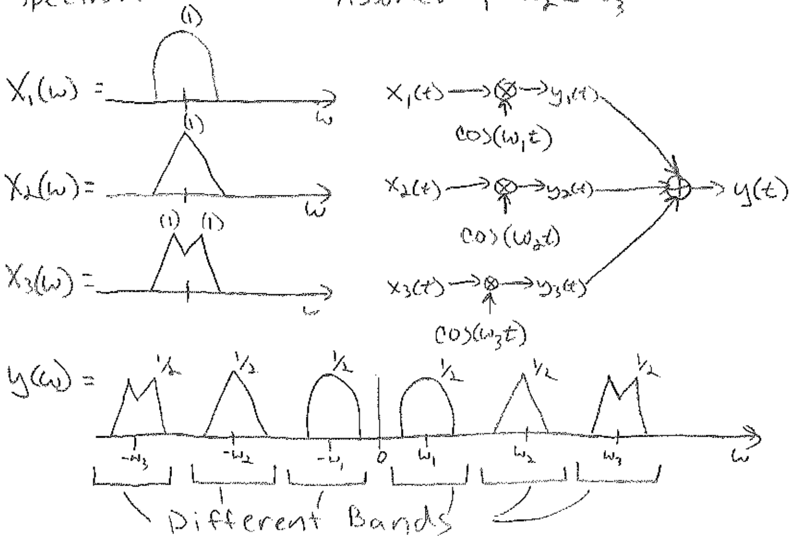




11/20-Lecture

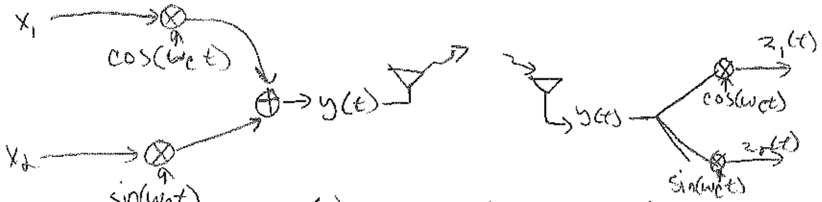
Frequency Division multiplexing:

spectrum Assume: $\omega_1 < \omega_2 < \omega_3$



If I'm interested in X_2 ($\overset{\Delta}{\text{triangle}}$) then multiply $\overset{\leftarrow\text{not divide}}{/}$ modulate $y(t)\cdot\cos(\omega_2 t) \rightarrow$ copies of Δ, then superimpose @ $\omega=0 \Rightarrow$ LPF to isolate it.

Another scheme allows you to modulate a pair of distinct signals X_1 & X_2 onto the same carrier frequency ω_c: Quadrature AM:



$y(t) = X_1\cos(\omega_c t) + X_2\sin(\omega_c t)$

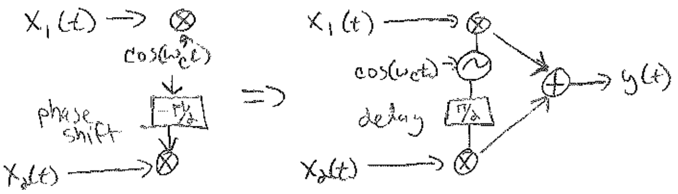
$z_1(t) = \frac{X_1(t)}{2}(1+\cos(2\omega_c t)) + \frac{X_2(t)}{2}\sin(\omega_c t)$

using various trig subs ↓

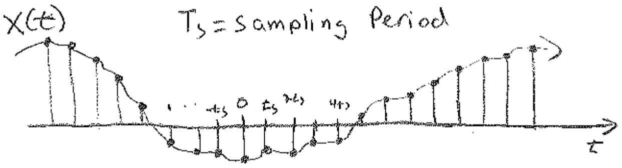
$z_1(t) = \frac{1}{2}X_1(t) + \frac{1}{2}X_1(t)\cos(2\omega_c t) + \frac{1}{2}X_2(t)\sin(\omega_c t)$

do similar trigonometic magic for $z_2(t)$

Use some sort of LPF to get "rid" of the unwanted frequencies.



Sampling theory:

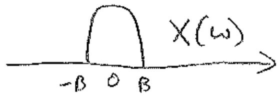


Time domain discretation? Can I recover X(t) from its samples?

Ans: a qualified "Yes"

Our signal space of Interest:

Band limited signals → closed universe

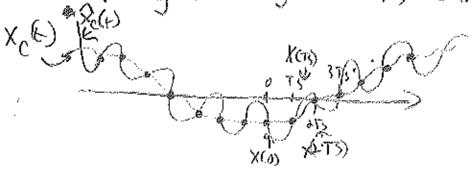


If I sample X @ a rate faster than 2B I can recover X from its samples.

$\omega_s = \frac{2\pi}{T_s}$ $\omega_s \geq 2B$

11/25 - Lecture

Sampling theory   Ts = sampling period $\omega_s = \frac{2\pi}{T_s}$



Q: can we recover x_c from the samples?

Yes, provided ...

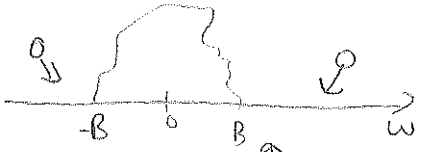
think of a time signal

$x_d(n) = x_c(nT_s)$

$$x_c(t) \rightarrow \boxed{C/D} \rightarrow x_d(n) = x_c(nT_s)$$
$$\uparrow T_s$$

convert an infinite set into a finite set and then recover the infinite set.

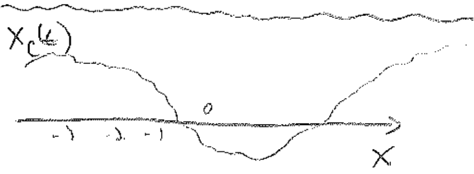
Ans: if x_c is band limited and we sample it fast enough, then we can recover x_c from x_d

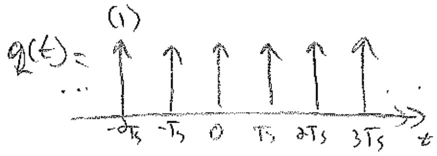


Band limited

cant vary faster than a certain amount.

$\omega_s \geq 2B$

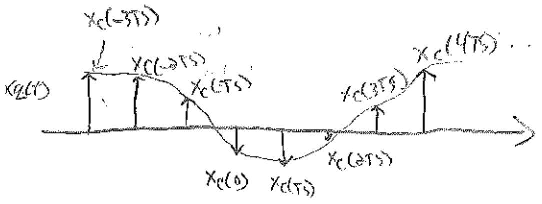




$x_q(t)$ = sequence of impulses $x_c(t) \circ q(t)$

$$x_q(t) \rightarrow \boxed{\text{Direct to Kroniker}} \rightarrow x_d(n)$$

this step loses NO information



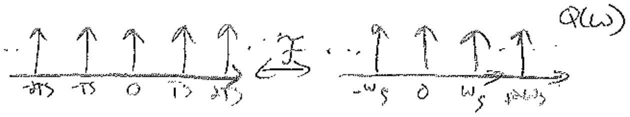
But $x(t) \rightarrow x_c(t) \cdot q(t)$

this can loose information.

Multiplication by the impulse train is potentially destructive:

$X_q(\omega) = \frac{1}{2\pi}(X_c * Q)(\omega)$

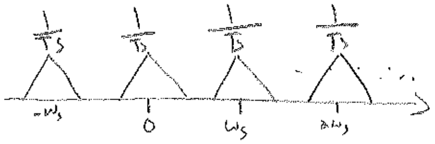
$q(t) = \sum_k \delta(t - kT_s)$



$q(t) = \sum_k Q_k e^{ik\omega_s t} = \frac{1}{T_s} \sum_k e^{ik\omega_s t}$

$e^{ik\omega_s t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - k\omega_s) \rightarrow Q(\omega) = \frac{2\pi}{T_s} \sum_k \delta(\omega - k\omega_s)$

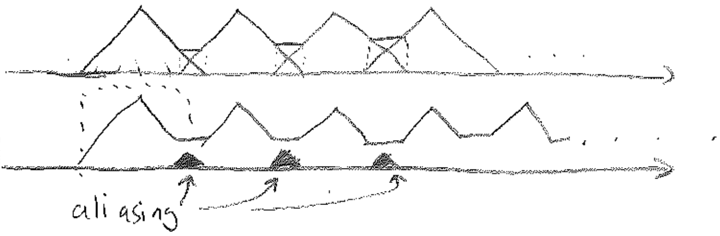
$X_q(\omega) = \frac{1}{2\pi}(X_c * Q)(\omega)$



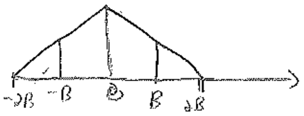
$\omega_s - B \geq B$

$\omega_s \geq 2B \rightarrow$ nyquist rate.

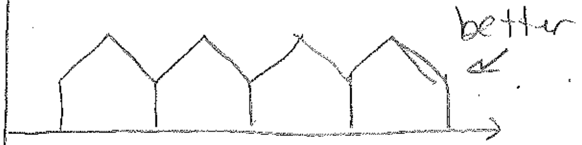
if we dont sample fast enough



Anti aliasing filter

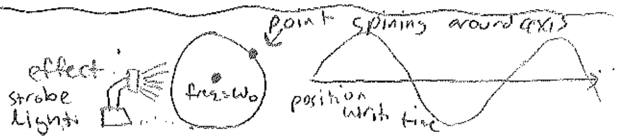


$$x_c(t) \rightarrow \boxed{\text{AA filter}} \rightarrow x_c'(t)$$



Carrage wheel effect

at ω_s freq:



if $\omega_s = \omega_0$ then dot is at same place every time.

$x_c(t) = e^{i\omega_0 t} \rightarrow$ position of dot at time t

sampling Therom: if $\omega_s \geq 2\omega_0 \Rightarrow$ we can recover

lets use $\omega_s = \frac{3}{2}\omega_0 \rightarrow$ Plot $X_q(\omega)$

$$x_c(t) \rightarrow \overset{q(t)}{\otimes} \rightarrow x_q(t) \frown\!\!\rightarrow X_q(\omega)$$