The Laplace Transform (Chapter 6)

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Definition of the Laplace Transform:

One of the most useful integral transforms for solving linear differential equations is the **Laplace Transform**, which is defined as (see Theorem 6.1.2 for conditions of existence):

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

The Laplace transform is particularly useful when dealing with problems that involve piecewise continuous functions, as we will see later.

Theorem 6.1.1:

(This theorem is essentially a restatement of convergence and divergence conditions of improper integrals when compared to each other, as seen in Calculus II. These are important in determining what functions the Laplace transform will exist for, because the definition of the transform involves an improper integral that needs to converge.)

Uniqueness of the Laplace Transform:

The Laplace transform is unique or one-to-one. That is:

$$\mathcal{L}(f_1) = \mathcal{L}(f_2) \Rightarrow f_1 = f_2$$

The proof of the uniqueness of the Laplace transform is beyond the scope of our textbook, as well as this presentation. The rigorous proof for the Laplace transform is actually a formal proof for a transform more general than the Laplace transform, and thus requires theorems from complex number theory. However, a more specific, simpler, and less rigorous proof is attached, taken from www.mit.edu.

<u>Theorem 6.1.2:</u>

Suppose that the following two statements are true:

1. f is a piecewise continuous on the interval $0 \leq t \leq A \quad \forall A > 0$
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$, where $K, M > 0$, and

Then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

Example 1: $\boldsymbol{f(t) = 1}$

Find the Laplace of $f(t) = 1, t \geq 0$.

We apply the definition of the Laplace transform:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt = \boldsymbol{\frac{1}{s}, s > 0}$$

Example 2: $\boldsymbol{f(t) = e^{at}}$

Find the Laplace of $f(t) = e^{at}, t \geq 0$.

Again, we apply the definition:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \boldsymbol{\frac{1}{s-a}, s > a}$$

It should be noted that the Laplace Transform is linear, or

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

This can be shown by applying the definition of the Laplace Transform to expand the equation and using the linear properties of integrals. Therefore, we can see that if Example 2 were modified to include a constant c, the Laplace of such a function will be the result of Example 2 multiplied by that same c. We will later use this result in Example 3 to find an inverse transform.

Other useful Laplace Transforms (derivation not shown)

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0, n \in \mathbb{N}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, \quad s > 0$$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \quad n \in \mathbb{N}$$

These three transforms are taken from table 6.2.1 from pg 304 of our Boyce DiPrima Differential Equations seventh edition textbook. The full table follows:

Theorem 6.2.1:

Suppose that f is continuous and f' is piecewise continuous on the interval $0 \leq t \leq A$, and $\exists K, a, M$ such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

is true.

Proof of Theorem 6.2.1:

Consider the integral

$$\int_0^A e^{-st} f'(t)\, dt$$

If f' has any discontinuous points in the interval $0 \leq t \leq A$, $t_1, t_1, \ldots, t_n$, then we can rewrite this integral as

$$\int_0^A e^{-st} f'(t)\, dt = \int_0^{t_1} e^{-st} f'(t) + \int_{t_1}^{t_2} e^{-st} f'(t)\, dt + \cdots + \int_{t_n}^{A} e^{-st} f'(t)\, dt$$

because of the linear property of integrals. Then we can integrate each term on the right side of the equations via integration by parts

$$\int_0^A e^{-st} f'(t)\, dt$$
$$= e^{-st} f(t)\big|_0^{t_1} + e^{-st} f(t)\big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\big|_{t_n}^{A} + s\left[\int_0^{t_1} e^{-st} f(t) + \int_{t_1}^{t_2} e^{-st} f(t)\, dt + \cdots + \int_{t_n}^{A} e^{-st} f(t)\, dt\right]$$

Since f is continuous, we can combine the integrated terms, giving

$$\int_0^A e^{-st} f'(t)\, dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\, dt$$

Now take the limit as $A \to \infty$. For $s > a$, we have $e^{-sA} f(A) \to 0$. So for $s > a$, we have

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

which was what we set out to prove. Note that this has similar implications for $\mathcal{L}\{f''(t)\}$, $\mathcal{L}\{f'''(t)\}$, and so on. In particular, the Laplace transform of f'' will be useful, as we will see in Example 3. We will prove the following statement of $\mathcal{L}\{f''(t)\}$.

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

Proof:

This can be easily found by directly using the result of Theorem 6.2.1:

$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

We can also conclude a corollary from Theorem 6.2.1:

Corollary 6.2.2

Supposed that the functions $f, f', \dots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants $K, a,$ and M such that $|f(t)| \leq Ke^{at}, \left|f'^{(t)}\right| \leq Ke^{at}, \dots, \left|f^{(n-1)}(t)\right| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f^{(n)}\}$ exists for $s > a$ and is given by:

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

The proof of this corollary is obtained by continually applying the result of Theorem 6.2.1 an n number of times.

Example 3: Application of the Laplace transform to an ODE with constant coefficients

(Problem #12 on pg 307)

Use the Laplace transform to solve the given initial value problem:

$y'' + 3y' + 2y = 0; \qquad y(0) = 1, \quad y'(0) = 0$

First, we take the Laplace transform of the ODE:

$$\mathcal{L}\{y'' + 3y' + 2y\} \\ = \mathcal{L}\{y''\} + \mathcal{L}\{3y'\} + \mathcal{L}\{2y\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3[s\mathcal{L}\{y\} - y(0)] \\ + 2\mathcal{L}\{y\} = 0$$

$= s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = 0$

We then apply the initial conditions.

$= s^2Y(s) - s - 0 + 3[sY(s) - 1] + 2Y(s) = 0$

$= s^2Y(s) + 3sY(s) + 2Y(s) - s - 3 = 0$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s+3}{s^2 + 3s + 2}$$

Then by partial fractions, we can further simplify this expression

$$Y(s) = \frac{s+3}{s^2 + 3s + 2} = \frac{2s - 4 - s + 1}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

We now must take the inverse Laplace to get our final solution. We use the result of Example 2

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a$$

along with our knowledge that the Laplace transform is linear to get

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boldsymbol{2e^{-t} - e^{-2t}}$$

Application of Laplace transform to a problem involving variable coefficients?

In general, attempting to solve a differential equation with even simple polynomial coefficients can prove to be ineffective. Consider

$$\mathcal{L}\{f(t)\cdot g(t)\} = \int_0^\infty e^{-st} f(t)g(t)\, dt$$

We can see that evaluating the integral on the right-hand side in general is troublesome, as the integrand involves not two, but three functions multiplied by each other. Therefore, no general solution can be found. For this reason, using the Laplace transform to solve differential equations involving variable coefficients is not recommended. If possible, another method should be used.

Step Functions:

The Laplace Transform is best suited for problems involving the unit step function. We have already done an example on how to apply the Laplace transform to a simple initial value problem. But how is the Laplace transform any more suitable for solving piecewise continuous functions than other methods? The key to utilizing the Laplace transform in such problems is the **unit step function,** also known as the **Heaviside function**, which is defined as:

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \quad c \geq 0$$

Note that the graph of $u_c(t)$ is discontinuous at one point: $t = c$. The Laplace transform of the unit step function is easily calculated:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \boldsymbol{\frac{e^{-cs}}{s}, s > 0}$$

It would be useful to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

which represents a translation of f a distance c in the positive t direction. We can rewrite $g(t)$ in terms of the unit step function:

$$g(t) = u_c(t)f(t-c)$$

Theorem 6.3.1:

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

In other words, Theorem 6.3.1 states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1, we compute $\mathcal{L}\{u_c(t)f(t-c)\}$:

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt$$

We now substitute a new integration variable $p = t - c$

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-(p+c)s}f(p)\,dp = e^{-cs}\int_c^\infty e^{-sp}f(p)\,dp = e^{-cs}F(s)$$

Thus, the we have the first part of Theorem 6.3.1. By taking the inverse transform of both sides, we have the second part.

Example 4: Simple example confirmation of Theorem 6.3.1

Take $f(t) = 1$. We already know that

$$\mathcal{L}\{1\} = \frac{1}{s}$$

and

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

We can see that $\mathcal{L}\{u_c(t)\} = e^{-cs}\mathcal{L}\{1\}$.

Example 5: Applying Laplace to a simple piecewise continuous non-homogenous differential equation

Find the solution of the differential equation

$$2y'' + 2y' + 2y = g(t)$$

where $g(t) = u_5 - u_{20} = \begin{cases} 1, & 5 \leq t < 20 \\ 0, & 0 \leq t < 5 \;\; and \;\; t \geq 20 \end{cases}$

with the initial conditions

$$y(0) = 0, \qquad y'(0) = 0$$

The Laplace transform of the first equation is

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = \frac{e^{-5s} - e^{-20s}}{s}$$

By substituting the initial values and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

To find $y = \phi(t)$, we rewrite $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s)$$

Where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

Then, let $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

To determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}$$

It can be found that $a = \frac{1}{2}, b = -1$, and $c = -\frac{1}{2}$. Thus we have,

$$H(s) = \frac{\frac{1}{2}}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{\frac{1}{2}}{s} - \frac{1}{2}\frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}$$

Which we can use Table 6.2.1 to obtain:

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos(\frac{\sqrt{15}t}{4}) + \frac{\sqrt{15}}{15}e^{-t/4}\sin(\frac{\sqrt{15}}{4})\right]$$

For $0 < t < 5$, the differential equation is

$$2y'' + y' + 2y = 0$$

By graphing the solution, we can see that for $0 < t < 5, y = 0$. We can then calculate the initial conditions at t approaches 5 from below:

$$y(5) = 0, \qquad y'(5) = 0$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1$$

whose solution we can obtain by plugging $h(t)$ back into the general form of our inverse transform. If we were to graph this solution, we would see that the solution is continuous, even at points $t = 5$ and $t = 20$ (see Figure 1). Again, we approximate the value and the derivative of the solution at $t = 20$, our next point of discontinuity

$$y(20) \cong 0.50161, \qquad y'(20) \cong 0.01124$$

From here, we have fully found all three parts of our solution.

At this point, you may be wondering why we don't simply evaluate the IVP as three separate problems. By using the Laplace transform, we can deal with points of discontinuity more elegantly and conveniently than if we used other methods.

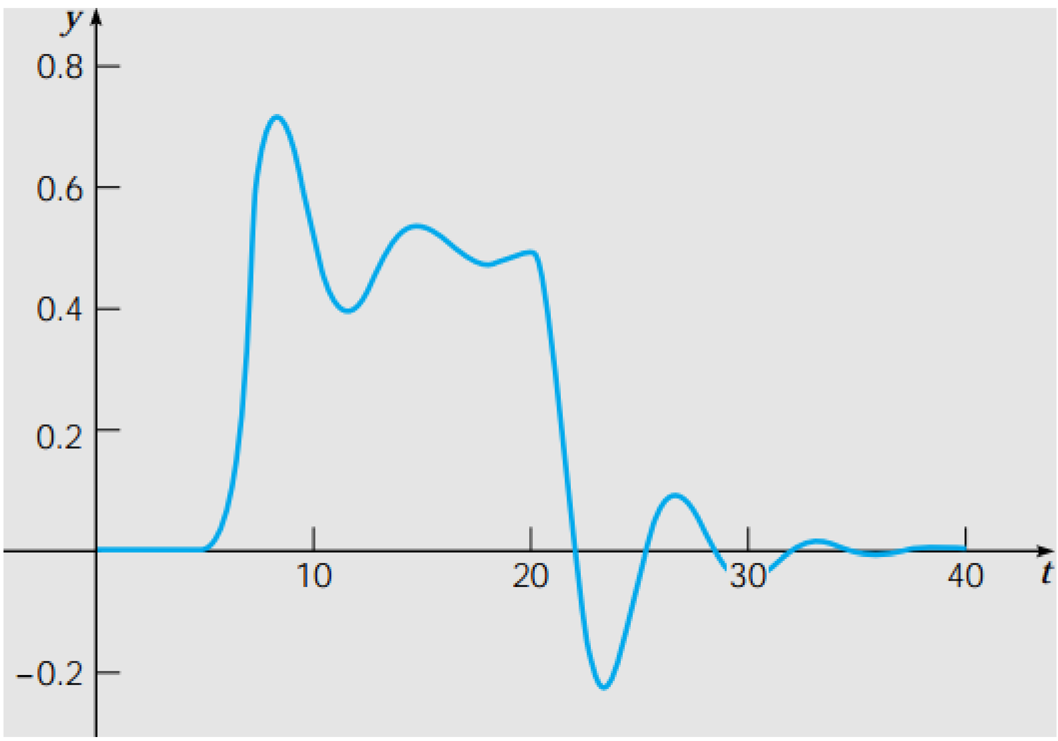

FIGURE 1 Graph of the solution to Example 5.

Further Implications

In conclusion, the Laplace transform is especially useful when dealing with problems that involve piecewise discontinuous functions. Furthermore, the Laplace transform is useful for dealing with phenomena of an impulsive nature, for example, voltages or forces that act with a large magnitude over very short time intervals. For further reading on the Laplace transform, we should familiarize ourselves with topics such as the **impulse function** or the **convolution integral**, which helps identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$.