CH 6: INNER-PRODUCT SPACES

norm: length of a vector, given by $\| \boldsymbol{x} \| = \sqrt{x_1^2 + \cdots + x_n^2}$
*note: norm is *not* linear on $\mathbb{R}^n$

dot product: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \cdots + x_n y_n$

inner product on V: a function that takes each ordered pair (u, v) of elements of V to a number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \in F$ and obeys the following:
1. positivity: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0 \; \forall \, \boldsymbol{v} \in V$
2. definiteness: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \iff \boldsymbol{v} = 0$
3. additivity in first slot: $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle \; \forall \, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$
4. homogeneity in first slot: $\langle a\boldsymbol{v}, \boldsymbol{w} \rangle = a\langle \boldsymbol{v}, \boldsymbol{w} \rangle \; \forall \, a \in F; \; \boldsymbol{v}, \boldsymbol{w} \in V$
5. conjugate symmetry: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{w}, \boldsymbol{v} \rangle} \; \forall \, \boldsymbol{v}, \boldsymbol{w} \in V$
*note: for $V = \mathbb{R}$, property 5 is simply $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$

the inner product gives notions of:
1. length: $\| \boldsymbol{v} \|$
2. angle θ between two vectors: $\cos \theta = \frac{\langle x|y \rangle}{\|x\|\|y\|}$

inner product space: a vector space V with an inner product on V

norm (revisited): for fin dim $\langle | \rangle$ V, can define as $\| \boldsymbol{x} \| = \sqrt{\langle \boldsymbol{v} | \boldsymbol{v} \rangle}$

NOTATION: from now on, V represents fin dim inner product space

orthogonal: describes two vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ for which $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$

orthogonal projection of $\boldsymbol{u}$ $\boldsymbol{onto}$ $\boldsymbol{span}(\boldsymbol{v})$: $\frac{\langle u|v \rangle}{\|v\|^2} \cdot \boldsymbol{v}$

pythagorean theorem: if $\boldsymbol{u}, \boldsymbol{v} \in V$ orthogonal in fin dim $\langle | \rangle$ V, then $\| \boldsymbol{u} + \boldsymbol{v} \|^2 = \| \boldsymbol{u} \|^2 + \| \boldsymbol{v} \|^2$

cauchy-schwarz inequality: if $\boldsymbol{u}, \boldsymbol{v} \in V$, then $|\langle \boldsymbol{u} | \boldsymbol{v} \rangle| \leq \| \boldsymbol{u} \| + \| \boldsymbol{v} \|$
*note: $= \iff a\boldsymbol{u} = \boldsymbol{v} \; or \; a\boldsymbol{v} = \boldsymbol{u} \; for \; some \; a \in F$

triangle inequality: if $\boldsymbol{u}, \boldsymbol{v} \in V$, then $\| \boldsymbol{u} + \boldsymbol{v} \| \leq \| \boldsymbol{u} \| + \| \boldsymbol{v} \|$
*note: $= \iff a\boldsymbol{u} = \boldsymbol{v} \; or \; a\boldsymbol{v} = \boldsymbol{u} \; for \; some \; a \geq 0 \in F$

parallelogram inequality: if $\boldsymbol{u}, \boldsymbol{v} \in V$, then $\| \boldsymbol{u} + \boldsymbol{v} \|^2 + \| \boldsymbol{u} - \boldsymbol{v} \|^2 = 2(\| \boldsymbol{u} \|^2 + \| \boldsymbol{v} \|^2)$

orthonormal: describes a list of vectors if the vectors in it are pair-wise orthogonal and each vector has norm 1

prop 6.15: if $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in V, then $\| a_1 \boldsymbol{e_1} + \cdots + a_m \boldsymbol{e_m} \|^2 = |a_1|^2 + \cdots + |a_m|^2 \; \forall \, a_1, \ldots, a_m \in F$

cor 6.16: every orthonormal list of vectors is linearly independent

orthonormal basis: an orthonormal list of vectors in V that is also a basis of V

thm 6.17: suppose $(\boldsymbol{e_1}, \ldots, \boldsymbol{e_n})$ is an orthonormal basis of V. then:
1. $\boldsymbol{v} = \langle \boldsymbol{v} | \boldsymbol{e_1} \rangle \boldsymbol{e_1} + \cdots + \langle \boldsymbol{v} | \boldsymbol{e_n} \rangle \boldsymbol{e_n}$
2. $\| \boldsymbol{v} \|^2 = |\langle \boldsymbol{v} | \boldsymbol{e_1} \rangle|^2 + \cdots + |\langle \boldsymbol{v} | \boldsymbol{e_n} \rangle|^2$

gram-schmidt procedure: if $(\boldsymbol{v_1}, \ldots, \boldsymbol{v_m})$ is a linearly independent list of vectors in V, then $\exists$ orthonormal list $(\boldsymbol{e_1}, \ldots, \boldsymbol{e_m})$ of vectors in V such that $span(\boldsymbol{v_1}, \ldots, \boldsymbol{v_{j-1}}) = span(\boldsymbol{e_1}, \ldots, \boldsymbol{e_j})$ for j=1,…,m

cor 6.24: every fin dim inner product space has an orthonormal basis

cor 6.25: every orthonormal list of vectors in V can be extended to an orthonormal basis of V

cor 6.27: suppose $T \in \mathcal{L}(V)$, if T has upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some orthonormal basis of V

cor 6.28: if V fin dim complex and $T \in \mathcal{L}(V)$, then T has upper-triangular matrix with respect to some orthonormal basis of V

orthogonal complement: the set of all vectors that are orthogonal to every vector in $U \subset V$, denoted by $U^\perp = \{\boldsymbol{v} \in V : \langle \boldsymbol{v} | \boldsymbol{u} \rangle = 0 \; \forall \, \boldsymbol{u} \in U\}$

thm 6.29: if $U \subset V$ a subspace, then $V = U \oplus U^\perp$

cor 6.33: if $U \subset V$ a subspace, then $U = (U^\perp)^\perp$

for $U \subset V$ a subspace, $V = U \oplus U^\perp \implies$ each $\boldsymbol{v} \in V$ can be written uniquely as $\boldsymbol{v} = \boldsymbol{u} \in U + \boldsymbol{w} \in U^\perp$. here, $\boldsymbol{u}$ is the **orthogonal projection** of V onto U, denoted by P_U, such that $P_U \boldsymbol{v} = \boldsymbol{u}$, and it obeys the following:
1. $range(P_U) = U$
2. $null(P_U) = U^\perp$
3. $v - P_U v \in U^\perp \; \forall \, \boldsymbol{v} \in V$
4. $P_U^2 = P_U$
5. $\| P_U \boldsymbol{v} \| \leq \| \boldsymbol{v} \| \;\; \forall \, \boldsymbol{v} \in V$

prop 6.36: suppose $U \subset V$ a subspace and $\boldsymbol{v} \in V$, then $\| \boldsymbol{v} - P_U \boldsymbol{v} \| \leq \| \boldsymbol{v} - \boldsymbol{u} \| \;\; \forall \, \boldsymbol{u} \in U$

linear functional: a linear map from V to the scalars

thm 6.45: suppose φ a linear functional on V, then $\exists! \, \boldsymbol{v} \in V$ such that $\varphi(\boldsymbol{u}) = \langle \boldsymbol{u} | \boldsymbol{v} \rangle \; \forall \, \boldsymbol{u} \in V$

NOTATION: W is fin dim nonzero inner product space over F

adjoint: for $T \in \mathcal{L}(V, W)$, the function T* from W to V such that $\langle T\boldsymbol{v} | \boldsymbol{w} \rangle = \langle \boldsymbol{v} | T^* \boldsymbol{w} \rangle \; \forall \, \boldsymbol{v} \in V$

the function $T \to T^*$ obeys the following $\forall \, S, T \in \mathcal{L}(V, W)$:
1. additivity: $(S + T)^* = S^* + T^*$
2. conjugate homogeneity: $(aT)^* = \bar{a} T^* \; \forall \, a \in F$
3. adjoint of adjoint: $(T^*)^* = T$
4. identity: $I^* = I$
5. products: $(ST)^* = T^* S^*$

prop 6.46: suppose $T \in \mathcal{L}(V, W)$, then
1. $null(T^*) = (rangeT)^\perp$
2. $range(T^*) = (nullT)^\perp$
3. $null(T) = (rangeT^*)^\perp$
4. $range(T) = (nullT^*)^\perp$

to obtain the nxm **conjugate transpose** of an mxn matrix, interchange the rows and columns of the latter and then take the complex conjugate of each entry.

prop 6.47: suppose $T \in \mathcal{L}(V, W)$, if $(\boldsymbol{e_1}, \ldots, \boldsymbol{e_n})$ is an orthonormal basis of V and $(\boldsymbol{f_1}, \ldots, \boldsymbol{f_m})$ is an orthonormal basis of W, then $\mathcal{M}(T^*, (\boldsymbol{f_1}, \ldots, \boldsymbol{f_m}), (\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}))$ is the conjugate transpose of $\mathcal{M}(T, (\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}), (\boldsymbol{f_1}, \ldots, \boldsymbol{f_m}))$

CH 7: OPERATORS ON INNER-PRODUCT SPACES

self-adjoint: describes an operator T if $T = T^*$

prop 7.1: every eigenvalue of a self-adjoint operator is real

prop 7.2: if V complex inner product and T operator on V such that $\langle T\boldsymbol{v}|\boldsymbol{v}\rangle = 0 \; \forall \; \boldsymbol{v} \in V$, then $T = 0$

cor 7.3: suppose V complex inner product and let $T \in \mathcal{L}(V)$, then T self-adjoint on V $\Leftrightarrow$ $\langle T\boldsymbol{v}|\boldsymbol{v}\rangle \in \mathbb{R} \; \forall \; \boldsymbol{v} \in V$

prop 7.4: if T self-adjoint on V such that $\langle T\boldsymbol{v}|\boldsymbol{v}\rangle = 0 \; \forall \; \boldsymbol{v} \in V$, then $T = 0$

normal: describes an operator that commutes with its adjoint, T*T=TT*

prop 7.6: an operator $T \in \mathcal{L}(V)$ is normal $\Leftrightarrow$ $\|T\boldsymbol{v}\| = \|T^*\boldsymbol{v}\| \; \forall \; \boldsymbol{v} \in V$

cor 7.7: suppose $T \in \mathcal{L}(V)$ normal, if $\boldsymbol{v} \in V$ an eigenvector of T with eigenvalue $\lambda \in F$, then $\boldsymbol{v}$ also an eigenvector of T* with eigenvalue $\bar{\lambda}$

cor 7.8: if $T \in \mathcal{L}(V)$ normal, then eigenvectors of T corresponding to distinct eigenvalues are orthogonal

complex spectral thm: suppose V complex inner product and $T \in \mathcal{L}(V)$, then V has an orthonormal basis consisting of eigenvectors of $T \Leftrightarrow T$ normal

lemma 7.11: suppose $T \in \mathcal{L}(V)$ self-adjoint, if $\alpha, \beta \in \mathbb{R}$ are such that $\alpha^2 < 4\beta$, then $T^2 + \alpha T + \beta I$ invertible

lemma 7.12: suppose $T \in \mathcal{L}(V)$ self-adjoint, then T has an eigenvalue

real spectral thm: suppose V real inner product and $T \in \mathcal{L}(V)$, then V has an orthonormal basis consisting of eigenvectors of $T \Leftrightarrow T$ self-adjoint

cor 7.14: suppose $T \in \mathcal{L}(V)$ self-adjoint and let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T, then $V = null(T - \lambda_1 I) \oplus \dots \oplus null(T - \lambda_m I)$ and each vector in each $null(T - \lambda_1 I)$ is orthogonal to all vectors in the other subspaces of this decomposition

lemma 7.15: suppose V two-dim real inner product and $T \in \mathcal{L}(V)$, then the following are equivalent:
1. T normal but not self-adjoint
2. matrix of T with respect to *every* orthonormal basis of V has form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $b \neq 0$
3. matrix of T with respect to *some* orthonormal basis of V has form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $b > 0$

prop 7.18: suppose $T \in \mathcal{L}(V)$ normal and $U \subset V$ a subspace invariant under T, then:
1. $U^\perp$ is invariant under T
2. U is invariant under T*
3. $(T|_U)^* = (T^*)|_U$
4. $T|_U$ is a normal operator on U
5. $T|_{U^\perp}$ is a normal operator on $U^\perp$

block diagonal matrix: a square matrix of form $\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$ where $A_1, \dots, A_m$ are square matrices lying along diagonal and all other entries equal 0

thm 7.25: suppose V real inner product and $T \in \mathcal{L}(V)$, then T normal $\Leftrightarrow$ $\exists$ orthonormal basis of V with respect to which T has a block diagonal matrix where each block is either a 1x1 matrix or a 2x2 matrix of form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $b > 0$

CH 8: OPERATORS ON COMPLEX VECTOR SPACES

suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$ an eigenvalue of T, then $\boldsymbol{v} \in V$ a **generalised eigenvector** of T corresponding to λ if $(T - \lambda I)^j \boldsymbol{v} = 0$ for some $j \in \mathbb{Z}^+$

prop 8.5: if $T \in \mathcal{L}(V)$ and $m \geq 0 \in \mathbb{Z}^+$ such that $nullT^m = nullT^{m+1}$, then $nullT^0 \subset nullT^1 \subset \dots \subset nullT^m = nullT^{m+1} = nullT^{m+2} = \cdots$

prop 8.6: if $T \in \mathcal{L}(V)$, then $nullT^{dimV} = nullT^{dimV+1} = nullT^{dimV+2} = \cdots$

cor 8.7: suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$ an eigenvalue of T, then the set of generalised eigenvectors of T corresponding to eigenvalue λ, denoted by $\tilde{E}_\lambda$, equals $null(T - \lambda I)^{dimV}$

nilpotent: describes an operator for which some power of it equals 0

cor 8.8: suppose $N \in \mathcal{L}(V)$ nilpotent, then $N^{dimV} = 0$

prop 8.9: if $T \in \mathcal{L}(V)$, then $rangeT^{dimV} = rangeT^{dimV+1} = rangeT^{dimV+2} = \cdots$

thm 8.10: let $T \in \mathcal{L}(V)$ and $\lambda \in F$, then for every basis of V with respect to which T has an upper-triangular matrix, λ appears on the diagonal of the matrix of T precisely dim $[null(T - \lambda I)^{dimV}]$ times

multiplicity: dimension of $\tilde{E}_\lambda$ for particular eigenvalue λ, which equals dim $[null(T - \lambda I)^{dimV}]$

prop 8.18: if V complex and $T \in \mathcal{L}(V)$, then the sum of the multiplicities of all the eigenvalues of T equals dimV

suppose V complex and $T \in \mathcal{L}(V)$. let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T, and d_j denote the multiplicity of λ_j as an eigenvalue. then the **characteristic polynomial** X_T of T is the polynomial $X_T = (z - \lambda_1)^{d_1} \dots (z - \lambda_m)^{d_m}$, which has degree dimV.

cayley-hamilton theorem: suppose V complex and $T \in \mathcal{L}(V)$ and let q denote X_T, then $q(T) = 0$

prop 8.22: if $T \in \mathcal{L}(V)$ and $p \in P(F)$, then $nullp(T)$ is invariant under T

thm 8.23: suppose V complex and $T \in \mathcal{L}(V)$, let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T and let $\tilde{E}_{\lambda_1}, \dots, \tilde{E}_{\lambda_m}$ be the corresponding subspaces of generalised eigenvectors, then
1. $V = \tilde{E}_{\lambda_1} \oplus \dots \oplus \tilde{E}_{\lambda_m}$
2. each $\tilde{E}_{\lambda_j}$ is invariant under T
3. each $(T - \lambda_j I)|_{\tilde{E}_{\lambda_j}}$ is nilpotent

CH 8: OPERATORS ON COMPLEX VECTOR SPACES (CONT.)

prop 8.18 restated: $dimV = dim\tilde{E}_{\lambda_1} + \cdots + dim\tilde{E}_{\lambda_m}$

cor 8.25: suppose V complex and $T \in \mathcal{L}(V)$, then ∃ basis of V consisting of generalised eigenvectors of T

lemma 8.26: suppose N a nilpotent operator on V, then ∃ basis of V with respect to which the matrix of N has form $\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$ where all entries on and below diagonal are 0

thm 8.28: suppose V complex and $T \in \mathcal{L}(V)$, let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T, then ∃ basis of V with respect to which T has a block diagonal matrix of form $\mathcal{M} = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$, where each A_j is an upper-triangular matrix of form $A_j = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$

minimal polynomial: the monic polynomial $p \in P(F)$ of smallest degree such that $p(T) = 0$

thm 8.34: let $T \in \mathcal{L}(V)$ and $q \in P(F)$, then $q(T) = 0 \Leftrightarrow$ minimal polynomial of T divides q

thm 8.36: let $T \in \mathcal{L}(V)$, then the roots of the minimal polynomial of T are precisely the eigenvalues of T

a kxk matrix is a **jordan block** if it is of form $A = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix}$. a block diagonal matrix $\mathcal{M}$ comprising jordan blocks is in **jordan canonical form**. then a basis of V is called a **jordan basis** for T if with respect to this basis, T has a block diagonal matrix in jordan form.

thm 8.47: suppose V complex, if $T \in \mathcal{L}(V)$, then ∃ basis of V that is a jordan basis for T