CH 1: VECTOR SPACES

list: $(x_1,\ldots,x_n)$ with finite length & defined order

vector space axioms:
1. **commutativity:** $u+v=v+u\ \forall\ u,v \in V$
2. **associativity:** $(u+v)+w=u+(v+w)\ \forall\ u,v,w \in V$
3. **additive identity:** $\exists\ 0 \in V$ such that $v+0=v\ \forall\ v \in V$
4. **additive inverse:** $\exists\ w \in V$ such that $v+w=0\ \forall\ v \in V$
5. **multiplicative identity:** $1v=v\ \forall\ v \in V$
6. **distributivity:** $a(u+v)=au+av$ & $(a+b)u=au+bu\ \forall\ u,v \in V,\ \forall\ a,b \in F$

P(F) ≡ set of all polynomials with coefficients in F where
1. $(p+q)(z)=p(z)+q(z)$
2. $(ap)(z)=ap(z)$

subspace: U is a subspace of V if
1. $0 \in U$
2. $u,v \in U \Rightarrow u+v \in U$ *(closed under addition)*
3. $a \in F,\ u \in U \Rightarrow au \in U$ *(closed under scalar multiplication)*

direct sum: $V=U_1\oplus\ldots\oplus U_n$ iff
1. $V=U_1+\ldots+U_n$
2. the only way to write 0 as a sum $u_1+\ldots+u_n$, where each $u_j \in U_j$, is to take all u_j's equal to 0

NOTE: 0 does *not* have a unique representation as a sum!

prop 1.9: suppose U,W subspaces of V, then $V=U\oplus W$ iff $V=U+W$ and $U\cap W=\{0\}$

CH 2: FINITE DIMENSIONAL VECTOR SPACES

span: set of all linear combinations of $(v_1+\ldots+v_m)$, $\text{span}(v_1+\ldots+v_m) = \{a_1v_1+\ldots+a_mv_m \mid a_1,\ldots,a_m \in U\}$

e.g., $(5,7,6)=\text{span}[(1,7,2),(2,0,2)]=(1,7,2)+2(2,0,2)$
nota: $\text{span}(v_1+\ldots+v_m)=V \Rightarrow$ "$(v_1+\ldots+v_m)$ spans V"

finite dimensional: a vector space that is spanned by some list of vectors in that space

infinite dimensional: a vector space that is spanned by no list of vectors in that space

linear independence: when the only choice of $a_1,\ldots,a_m \in F$ that makes $a_1v_1+\ldots+a_mv_m = 0$ is $a_1=\ldots=a_m=0$

i.e., each $v_j \in \text{span}(v_1,\ldots,v_m)$ has only 1 representation
e.g., $[(1,0,0),(0,1,0),(0,0,1)]$ is lin ind in F^4

linear dependence: when $\exists\ a_1,\ldots,a_m \in F$ not all 0 such that $a_1v_1+\ldots+a_mv_m=0$

e.g., $[(2,3,1),(1,-1,2),(7,3,8)]$ is lin dep in F^3
e.g., any list containing the 0 vector is lin dep

linear dependence lemma: if $(v_1+\ldots+v_m)$ is lin dep in V and $v_1 \neq 0$, then $\exists\ j \in \{2,\ldots,m\}$ such that
1. $v_j \in \text{span}(v_1,\ldots,v_m)$
2. if the j^{th} term is removed from $(v_1,\ldots,v_m)$, then the span of the remaining list equals $\text{span}(v_1,\ldots,v_m)$

thm 2.6: fin dim V, length of every lin ind list ≤ length of every spanning list

prop 2.7: every subspace of fin dim V is itself fin dim

basis: a list of vectors in V that is lin ind and spans V

standard basis of F^n: $[(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)]$

prop 2.8: a list $(v_1,\ldots,v_n) \in V$ is a basis of V iff every $v \in V$ can be written uniquely as $v=a_1v_1+\ldots+a_nv_n \mid a_1,\ldots a_n \in F$

thm 2.10: every spanning list in V can be reduced to a basis

cor 2.11: every fin dim V has a basis

thm 2.12: every lin ind list in fin dim V can be extended to a basis of V

prop 2.13: suppose fin dim V and U is a subspace of V, then $\exists$ a subspace W of V such that $V=U\oplus W$

thm 2.14: any two bases of fin dim V have the same length

dimension: the length of any basis of fin dim V

prop 2.15: if fin dim V, U subspace of V, then $\dim(U)\leq\dim(V)$

prop 2.16: if fin dim V, then every spanning list of vectors in V with length dim(V) is a basis of V

prop 2.17: if fin dim V, then every lin ind list of vectors in V with length dim(V) is a basis of V

thm 2.18: if U_1, U_2 are subspaces of fin dim V, then $\dim(U_1+U_2)=\dim(U_1)+\dim(U_2)-\dim(U_1\cap U_2)$

prop 2.19: suppose fin dim V and $U_1,\ldots,U_n$ are subspaces of V such that $V=U_1+\ldots+U_m$ and $\dim(V)=\dim(U_1)+\ldots+\dim(U_m)$, then $V=U_1\oplus\ldots\oplus U_m$

CH 3: LINEAR MAPS

linear map: a function $T:V\to W$ with properties
1. **additivity:** $T(u+v)=Tu+Tv\ \forall\ u,v \in V$
2. **homogeneity:** $T(av)=a(Tv)\ \forall\ v \in V,\ \forall\ a \in F$

types of linear maps
1. **zero:** $0 \in L(V,W) \equiv 0v=0$
2. **identity:** $I \in L(V,V) \equiv Iv=v$
3. **differentiation:** $T \in (P(R),P(R)) \equiv Tp=p'$
4. **integration:** $T \in (P(R),P(R)) \equiv Tp=\int p(x)dx,\ [0,1]$
5. **multiplication by x^2:** $T \in (P(R),P(R)) \equiv (Tp)(x)=x^2p(x),\ x \in R$
6. **backward shift:** $T \in (F^\infty,F^\infty) \equiv T(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)$
7. **from F^n to F^m:** $T \in (F^n,F^m) \equiv T(x_1,\ldots,x_n)=(a_{1,1}x_1+\ldots+a_{1,n}x_n+\ldots,a_{m,1}x_1+\ldots+a_{m,n}x_n)$

suppose $(v_1,\ldots,v_n)$ is a basis of V and $T:V\to W$ is linear. if $v \in V$, then we can write v as $v=a_1v_1+\ldots+a_nv_n$. linearity of T $\Rightarrow Tv=a_1Tv_1+\ldots+a_nTv_n$.

product: suppose $T \in L(U,V)$, $S \in L(V,W)$. then $ST \in L(U,W) \equiv (ST)v=S(Tv)$ for $v \in U$ with properties
1. **associativity:** $(T_1T_2)T_3=T_1(T_2T_3)$ whenever T_1,T_2,T_3 are linear maps whose products make sense
2. **identity:** $TI=T$ and $IT=T$ whenever $T \in L(V,W)$
3. **distributive:** $(S_1+S_2)T=S_1T+S_2T$ and $S(T_1+T_2)=ST_1+ST_2$ whenever $T,T_1,T_2 \in L(U,V)$ and $S,S_1,S_2 \in L(V,W)$

null space: for T ∈ L(V,W), subset of V consisting of those vectors that T maps to 0, null(T) = {v ∈ V | Tv=0}

e.g., for $(Tp)x=x^2p(x)$, null(T)={0}
e.g., for $T(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)$, null(T)={(a,0,0…) | a ∈ F}

prop 3.1: if T ∈ L(V,W), then null(T) is a subspace of V

injectivity: if whenever u,v ∈ V and Tu=Tv, we have u=v

prop 3.2: suppose T ∈ L(V,W), then T is inj iff null(T)={0}

range: for T ∈ L(V,W), subset of W consisting of those vectors of form Tv for some v ∈ V, range(T)={Tv | v ∈ V}

prop 3.3: if T ∈ L(V,W), then range(T) is a subspace of W

surjectivity: if for T ∈ L(V,W), range(T)=W

thm 3.4: if fin dim V and T ∈ L(V,W), then range(T) is fin dim subspace of W and dim(V)=dim[null(T)]+dim[range(T)]

cor 3.5: if fin dim V, fin dim W such that dim(V)>dim(W), then no linear map from V to W is inj

cor 3.6: if fin dim V, fin dim W such that dim(V)<dim(W), then no linear map from V to W is surj

calculating a matrix: let T ∈ L(V,W). suppose $(v_1,\ldots,v_n)$ is a basis of V and $(w_1,\ldots,w_m)$ is a basis of W. for each k=1,…,n, we can write Tv_k uniquely as a linear combination of w's:
$Tv_k=a_{1,k}w_1+\ldots+a_{m,k}w_m$ | aj,k ∈ F for j=1,…,m.
then matrix is given by $M(T,(v_1,\ldots,v_n),(w_1,\ldots,w_m))$, where basis vectors of domain are written across top and basis vectors of target space are written along left:

$$\begin{array}{c|ccccc} & v_1 & \ldots & v_k & \ldots & v_n \\ w_1 & [& & a_{1,k} & &] \\ \ldots & & & & & \\ w_m & [& & a_{m,k} & &] \end{array}$$

the k^{th} column consists of the scalars needed to write Tv_k has combination of w's

Tv_k is retrieved from M(T) by multiplying each entry in k^{th} column by the corresponding w from the left column, then adding up the resulting vectors

think of the k^{th} column as T applied to the k^{th} basis vector

e.g., if T ∈ (F^2,F^3) ≡ T(x,y)=(x+3y,2x+5y,7x+9y), then T(1,0)=(1,2,7) and T(0,1)=(3,5,9), so M(T) with respect to the standard bases is 3x2 matrix

$$M_B[T] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}$$

matrix properties
1. addition: M(T+S)=M(T)+M(S) whenever T ∈ L(V,W)
2. scalar multiplication: M(cT)=cM(T) T ∈ L(V,W), c ∈ F
3. matrix multiplication: M(TS)=M(T)M(S)

suppose $(v_1,\ldots,v_n)$ is a basis of V. if v ∈ V, then ∃! scalars $b_1,\ldots,b_n$ such that $v=b_1v_1+\ldots+b_nv_n$. then the **matrix** of v is the nx1 matrix defined by

$$M(v) = \begin{bmatrix} b_1 \\ \ldots \\ b_n \end{bmatrix}$$

prop 3.14: suppose T ∈ L(V,W) and $(v_1,\ldots,v_n)$ is a basis of V and $(w_1,\ldots,w_m)$ is a basis of W, then M(Tv)=M(T)M(v) for every v ∈ V

invertibility: for T ∈ L(V,W), if ∃ S ∈ L(W,V) such that $ST=I_V$ and $TS=I_W$

prop 3.17: a linear map is invertible iff it is inj & surj

isomorphism: a linear map T ∈ L(V,W) for which ∃ an invertible map from V onto W

thm 3.18: fin dim V, fin dim W are isomorphic iff dim(V)=dim(W)

thm 3.19: suppose $(v_1,\ldots,v_n)$ is a basis of V and $(w_1,\ldots,w_m)$ is a basis of W, then M is an invertible linear map between L(V,W) and Mat(m,n,F)

prop 3.20: if fin dim V, fin dim W, then fin dim L(V,W) and dim[L(V,W)]=dim(V)dim(W)

operator: a linear map from a vector space to itself

thm 3.21: suppose fin dim V, if T ∈ L(V,V), then
1. T is invertible
2. T is inj
3. T is surj

Ch 4: Polynomials

polynomial: a function p:F→F with coefficients $a_0,\ldots,a_m$ ∈ F such that $p(z)=a_0+a_1z+a_2z^2+\ldots+a_mz^m$ ∀ z ∈ F

a polynomial has **degree *m*** if $a_m \neq 0$

a polynomial has **degree ∞** if $a_1=\ldots=a_m=0$

root: a number λ ∈ F such that p(λ)=0

prop 4.1: suppose p ∈ P(F) is a polynomial with degree m≥1, then λ is a root of p iff ∃ q ∈ P(F) with degree m−1 such that p(z)=(z−λ)q(z) ∀ z ∈ F

cor 4.3: suppose p ∈ P(F) is a polynomial with degree m≥0, then p has ≥m distinct roots ∈ F

cor 4.4: suppose $a_0,\ldots,a_m$ ∈ F, if $a_0+a_1z+a_2z^2+\ldots+a_mz^m=0$ ∀ z ∈ F, then $a_1=\ldots=a_m=0$

division algorithm: suppose p,q ∈ P(F) with p≠0, then ∃ polynomials s,r ∈ P(F) such that q=sp+r and deg(r)<deg(p)

fundamental theorem of algebra: every nonconstant polynomial with complex coefficients has a root

cor 4.8: if p ∈ P(C) is a nonconstant polynomial, then p has a unique factorisation (except for the order of the factors) of the form $p(z)=c(z-\lambda_1)\ldots(z-\lambda_m)$, $c,\lambda_1,\ldots,\lambda_m$ ∈ C

suppose z=a+bi, where a,b ∈ R. then a=Re(z) is the real part and b=Im(z) is the imaginary part. so for any complex number z, z=Re(z)+Im(z)i.

complex conjugate: z*=Re(z)−Im(z)i

absolute value: $|z|=\sqrt{[Re(z)]^2+[Im(z)]^2}$

prop 4.10: suppose p is a polynomial with real coefficients, if $\lambda \in C$ is a root of p, then so is λ^*

prop 4.11: let $\alpha,\beta \in R$, then ∃ polynomial factorisation of the form $x^2+\alpha x+\beta=(x-\lambda_1)(x-\lambda_2)$ with $\lambda_1, \lambda_2 \in R$, iff $\alpha^2 \geq 4\beta$

thm 4.14: if $p \in P(R)$ is nonconstant, then p has unique factorisation (except for order of factors) of the form $p(x)=c(x-\lambda_1)\ldots(x-\lambda_m)(x^2+\alpha_1 x+\beta_1)\ldots(x^2+\alpha_M x+\beta_M)$ where $\lambda_1\ldots\lambda_m \in R$ and $(\alpha_1, \beta_1),\ldots,(\alpha_M,\beta_M) \in R^2$ with $\alpha_j<\beta_j$ for each j

Ch 5: Eigenvalues and Eigenvectors

invariance: if $u \in U$ implies $Tu \in U$

eigenvalue: a scalar $\lambda \in F$ for which ∃ $u \neq 0 \in V$ such that $Tu=\lambda u$ for $T \in L(V,V)$

$Tu=\lambda u$ is equivalent to $(T-\lambda I)u=0$, so λ is an eigenvalue of T iff $T-\lambda I$ is not injective. then by thm 3.21, λ is an eigenvalue iff $T-\lambda I$ is not invertible, which happens iff $T-\lambda I$ is not surjective.

eigenvector: a vector $u \neq 0 \in V$ such that $Tu=\lambda u$ for an eigenvalue $\lambda \in F$ and $T \in L(V,V)$

the set of eigenvectors of T corresponding to λ equals $null(T-\lambda I)$

thm 5.6: let $T \in L(V,V)$ and suppose $\lambda_1,\ldots,\lambda_m$ are distinct eigenvalues of T and $v_1,\ldots,v_m$ are corresponding nonzero eigenvectors, then $(v_1,\ldots,v_m)$ is lin ind

cor 5.9: each operator on V has ≥dim(V) distinct eigenvalues

thm 5.10: every operator on fin dim, nonzero, complex V has an eigenvalue

suppose $(v_1,\ldots,v_m)$ is a basis of V. for each $k=1,\ldots,n$, we can write $Tv_k=a_{1,k}v_1+\ldots+a_{n,k}v_n$, where $a_{j,k} \in F$ for $j=1,\ldots n$. the **matrix of T with respect to the basis $(v_1,\ldots,v_m)$** is

$$M_B[T]=\begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & & \ldots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix}$$

upper triangular: describes a matrix whose entries below the diagonal are 0

prop 5.12: suppose $T \in L(V,V)$ and $(v_1,\ldots,v_m)$ is a basis of V, then the following are equivalent

1. M[T] with respect to $(v_1,\ldots,v_m)$ is upper triangular
2. $Tv_k \in span(v_1,\ldots,v_k)$ for each $k=1,\ldots,n$
3. $span(v_1,\ldots,v_k)$ is invariant under T for each $k=1,\ldots,n$

thm 5.13: suppose V complex and $T \in L(V,V)$, then T has an upper triangular matrix with respect to some basis of V

prop 5.16: suppose $T \in L(V,V)$ has an upper triangular matrix with respect to some basis of V, then T invertible iff all entries on the diagonal of said matrix ≠ 0

prop 5.18: suppose $T \in L(V,V)$ has an upper triangular matrix with respect to some basis of V, then the eigenvalues of T consist of the entries of the diagonal

diagonal matrix: nxn matrix that is 0 everywhere except possibly along the diagonal

not every operator has a diagonal matrix with respect to some basis

e.g., $T \in L(C^2,C^2)$ defined by $T(w,z)=(z,0)$. $\lambda=0$ is the only eigenvalue, and its corresponding set of eigenvectors are 1-dim subspace $\{(w,0) \in C^2 \mid w \in C\}$. so there aren't enough lin ind eigenvectors of T to form a basis of C^2, which is 2-dim.

prop 5.20: if $T \in L(V,V)$ has dim(V) distinct eigenvalues, then T has diagonal matrix with respect to some basis of V

prop 5.21: suppose $T \in L(V,V)$ and let $\lambda_1,\ldots,\lambda_m$ denote the distinct eigenvalues of T, then the following are equivalent

1. T has diagonal matrix with respect to some basis of V
2. V has a basis consisting of eigenvectors of T
3. ∃ 1-dim subspaces $U_1,\ldots,U_n$ of V, each T-inv, such that $V=U_1\oplus\ldots\oplus U_n$
4. $V=null(T-\lambda_1 I)\oplus\ldots\oplus null(T-\lambda_m I)$
5. $dim(V)=dim[null(T-\lambda_1 I)]+\ldots+dim[null(T-\lambda_m I)]$

thm 5.24: every operator on fin dim, nonzero, real V has an invariant subspace of dimension 1 or 2

thm 5.26: every operator on an odd-dim real V has an eigenvalue