Math 121A Midterm 1 Study Guide

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(Page 1

Series

A series is a sum of a sequence.

Partial Series:

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Infinite Series:

$$S = \lim_{n\to\infty} S_n = a_1 + a_2 + \cdots + a_k + \cdots = \sum_{i=1}^{\infty} a_i$$

If S exists, the series converges, otherwise it diverges.

Remainder:

$$R_n = \sum_{k=n+1}^{\infty} a_k = S - S_n \text{ when } S \text{ exists.}$$

Tests of Convergence for Series

Preliminary Test

If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges

Integral Test

& is non-negative!

If $0 < a_{n+1} \leq a_n$ for $n \geq N$, then $\sum_n a_n$ converges iff $\int_{Q}^{\infty} a_n \, dn$ is finite.

Comparison Principle

Suppose a_n, b_n are sequences where for all n,

$0 \leq a_n \leq b_n$

Then if $\sum b_n$ converges, $\sum_n a_n$ converges

if $\sum a_n$ diverges, $\sum_n b_n$ diverges

Special Comparison Principle

If a_n, b_n are non-negative sequences & $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, then if $\sum b_n$ converges, $\sum a_n$ also converges.

Ratio Test

Defining $\rho_n = \left| \frac{a_{n+1}}{a_n} \right|$ and $\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $\rho < 1$, the series converges. If $\rho > 1$ the series diverges.
If $\rho = 1$, the series is inconclusive.

Alternating Series Test

If a_n is an alternating series (i.e., $\text{sign}(a_{n+1}) = -\text{sign}(a_n)$), $|a_{n+1}| \leq |a_n|$, and $\lim_{n\to\infty} a_n = 0$, then the series converges. (If $\sum |a_n|$ converges, the series is absolutely convergen[t])

Magnitudes

$\log n \ll n \ll n^2 \ll \ldots \ll 2^n \ll n!$ (where $f \ll g$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$)

Geometric Series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r} \quad \text{(iff } |r|<1\text{, otherwise undefined).}$$

Power Series

Def: $\sum_{n=0}^{\infty} a_n x^n$

Theorem: A power series:

1. Converges everywhere (that is, $\forall x$)
2. Converges for $x=0$ only
3. Converges when $|x|<R$ and diverges when $|x|>R$ ($R \equiv$ radius of convergence)

(the convergence of a power series when $|x|=R$ must be checked explicitly and, in general, is not symmetric).

Maclaurin / Taylor Series

The power series expansion of an analytic function about a number a is known as a Taylor Series. When $a=0$, this is known as a Maclaurin Series.

¶ → i.e., if $f(x)$ is analytic,

$$f(x-a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

Note: radius of convergence for power series depends upon a.

Common Maclaurin Series

			convergent for:
$\sin(x) =$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	(all x)
$\cos(x) =$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	(all x)
¶ → $e^x =$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	(all x)
$\ln(1+x) =$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1 < x \le 1)$
$(1+x)^p =$	$\sum_{n=0}^{\infty} \binom{p}{n} x^n$	$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$	$(\lvert x\rvert < 1)$

Error of Series Approximations

$$R_N(x) \equiv f(x) - \left(f(a) + (x-a)f'(a) + (x-a)^2 f''(a)\left(\frac{1}{2!}\right) + \cdots + (x-a)^N f^{(N)}(a)\left(\frac{1}{N!}\right) \right) = \frac{(x-a)^{N+1} f^{(N+1)}(c)}{(N+1)}, \quad c \in [a, x]$$

Special Cases:

Alternating Series (really not having to do w/ Taylor Series...)

$|R_N| \le |a_{N+1}|$ (Also recall $|S| \le a_1$ & $S = S_N + R_N$)

Decreasing Coefficients

¶ → If $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x|<1$ & $|a_{n+1}| < |a_n|$ for $n > N$, then

$$R_N(x) = \frac{a_{N+1} x^N}{1-|x|}$$

Asymptotic Notation

How do we write lower order terms we are not concerned w/ so that we can keep track of them?

Math 121A Midterm 1 Study Guide

"Little-oh" Notation

Given continuous functions $f(x)$ & $g(x)$, we say that $f(x) = o(g(x))$ as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ (e.g. $x^5 = o(x)$ as $x \to 0$, $x^4 = o(x^5)$ as $x \to \infty$)

"Big-Oh" Notation

Given continuous functions $f(x)$ & $g(x)$, we say that $f(x) = O(g(x))$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty$. (e.g. $x^2 = O(x^2)$ as $x \to 0$, $2\sin(x) = O(1)$ as $x \to \infty$).

Rules for Manipulation for Asymptotic Notation

1. If $c \in \mathbb{R}$ & $f(x) = o(g(x))$, then $cf(x) = o(g(x))$
2. If $f_1(x) = o(g_1(x))$ & $f_2(x) = o(g(x))$, $f_1(x)f_2(x) = o(g_1(x)g_2(x))$
3. If $f(x) = o(g(x))$ then $xf(x) = o(x \cdot g(x))$
4. If $\lim_{x\to 0} g(x) = 0$, then $\frac{1}{1+g(x)} = 1 - g(x) + o(g(x))$
5. $o(f(x)) + o(g(x)) = o(f(x) + g(x))$
6. $o(o(f(x))) = o(f(x))$

All(?) apply to "Big-Oh" notation.

Linear Algebra

Coordinates & Change of Bases

Notation: $\underline{x}$ is a vector. $[\underline{x}]$ is the coordinate column vector w.r.t. the standard basis. $[x]_B$ is the coordinate column vector w.r.t. the basis B. $T(\underline{x})$ is the result of a transformation T on a vector $\underline{x}$. The standard basis elements are denoted $\underline{e}_1, \underline{e}_2, \ldots \underline{e}_n$.

- $[T(\underline{x})] = [[T(\underline{e}_1)]\,[T(\underline{e}_2)]\,\cdots\,[T(\underline{e}_n)]][\underline{x}] = [T][\underline{x}]$
- $[T(x)]_B = [[T(\underline{b}_1)]_B\,[T(\underline{b}_2)]_B\,\cdots\,[T(\underline{b}_n)]_B] = [T]_B[\underline{x}]_B$
- $[T] = B[T]_B B^{-1} \Leftrightarrow [T]_B = B^{-1}[T]B$

Diagonalization

- To Diagonalize a matrix is to express it as $A = CDC^{-1}$ where C is an invertable matrix & D is a diagonal matrix.
- Theorem: If A is symmetric ($A = A^T$)
  1. It's diagonalizable
  2. Its basis is orthogonal
  3. All eigenvalues are real
- Orthogonal Matrices
  1. All columns are orthogonal
  2. Their transpose = their inverse (i.e. if A is orthogonal, $A^T = A^{-1}$)
- Def'n: the adjoint of a matrix A: $A^\dagger = (A^T)^*$ (pronounced "A dagger")
- Theorem: If A is Hermetian ($A = A^\dagger$)
  1. A is diagonalizable as $A = UDU^\dagger$, where U is a unitary ($U^{-1} = U^\dagger$) matrix.
  2. A has an orthogonal eigen basis
  3. Its eigenvalues are real

Inner Product Spaces

Def'n: $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$ is called an inner product if:

1. $\langle \underline{x} | \underline{x} \rangle \geq 0$ & $= 0$ iff $\underline{x} = \underline{0}$
2. $\langle a_1 \underline{x}_1 + a_2 \underline{x}_2 | \underline{y} \rangle = a_1^* \langle \underline{x}_1 | \underline{y} \rangle + a_2^* \langle \underline{x}_2 | \underline{y} \rangle$
3. $\langle \underline{x} | b_1 \underline{y}_1 + b_2 \underline{y}_2 \rangle = b_1 \langle \underline{x} | \underline{y}_1 \rangle + b_2 \langle \underline{x} | \underline{y}_2 \rangle$
4. $\langle \underline{x} | \underline{y} \rangle = \langle \underline{y} | \underline{x} \rangle^*$

This in turn yields:

The Cauchy-Schwarz Inequality

$\langle \underline{p} | \underline{q} \rangle \leq \|\underline{p}\| \cdot \|\underline{q}\|$ (where $\|\underline{x}\| = \sqrt{\langle \underline{x} | \underline{x} \rangle}$ is the norm of $\underline{x}$).

Triangle Inequality

$\|\underline{p} + \underline{q}\| \leq \|\underline{p}\| + \|\underline{q}\|$

Pythagorean Theorem

$\langle \underline{p} | \underline{q} \rangle = 0 \Rightarrow \|\underline{p}\|^2 + \|\underline{q}\|^2 = \|\underline{p} + \underline{q}\|^2$

Partial Differentiation

- Def'n: $\frac{\partial f}{\partial x_i}(c_1, \dots, c_i, \dots, c_n) = \lim_{\Delta x_i \to 0} \frac{f(c_1, \dots, c_{i-1}, c_i + \Delta x_i, c_{i+1}, \dots, c_n) - f(c_1, \dots, c_{i-1}, c_i, c_{i+1}, \dots c_n)}{\Delta x_i}$
- A func. is differentiable at a point if it is well approximated by a linear function in the neighborhood of that point.
- Let's say $z = f(x, y)$. Then $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \Delta y + o(\Delta x) + o(\Delta y)$
- Then the total differential is the limit of this:

$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Notation

$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$, but what if $z = f(x, y)$ and $z = g(x, \theta)$, what is $\frac{\partial z}{\partial x}$? We can use the notation $\left(\frac{\partial z}{\partial x}\right)_y$ to denote "differentiating z w.r.t. x holding y constant." (dF_z is the differential dF holding z constant.)

Linear Approximations

E.g.: Find $\sqrt{(5.01)^2 - (3.98)^2}$

$f(x, y) = \sqrt{x^2 - y^2} \Rightarrow f(5, 4) = 3$

$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \Rightarrow \Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2 - y^2}} \Rightarrow \frac{\partial f}{\partial x}(5, 4) = \frac{5}{3}$

$\Rightarrow \frac{\partial f}{\partial y} = -\frac{2y}{2\sqrt{x^2 - y^2}} \Rightarrow \frac{\partial f}{\partial y}(5, 4) = -\frac{4}{3}$

$\Delta z = \frac{5}{3}(.01) + \left(-\frac{4}{3}\right)(-.02) = \frac{.13}{3} = .04333\ldots \Rightarrow \sqrt{(5.01)^2 - (3.98)^2} \approx 3 + .04333\ldots = 3.04333\ldots$

Implicit Partial Differentiation

E.g.: $x^2 + y^2 + z^2 = 1$; $\frac{\partial z}{\partial x} = ?$

$F(x, y, z) = x^2 + y^2 + z^2 - 1 \Rightarrow dF = 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow \left(\frac{dz}{dx}\right)_y = -\frac{x}{z}$

Chain Rule

E.g.: $z = f(x, y)$, $x = x(s, t)$, $y = y(s, t)$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$; $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

Minimization

- Extrema occur when $\nabla f = \underline{0}$. To check if maxima/minima, use the 2^{nd} Derivative Test. You can also use Lagrange Multipliers to find extrema when there is a constant on the domain.

2^{nd} Derivative Test

- Let $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ evaluated at a critical pt. (x_0, y_0).
- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$, then (x_0, y_0) is a <u>local min.</u>
- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$, then (x_0, y_0) is a <u>local max.</u>
- If $D < 0$, (x_0, y_0) is a saddle point.
- If $D = 0$, the test is inconclusive.

Lagrange Multipliers

- Let's say we need to find extrema of some function f while being subject to some constraint $g = c$ (where c is a const.)
- Method: 1 Solve $\{\nabla f = \lambda \nabla g,\ g = c\}$ 2 Do 2^{nd} Test on critical points.