Math 121A Midterm 2 Study Sheet

Harrison Bachrach

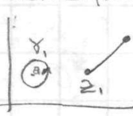
$i^2 \equiv -1$. $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$. $z = x + iy \Rightarrow Re(z) = x$, $Im(z) = y$

$z^{-1} = \frac{\bar{z}}{|z|}$, $\bar{z} = x - iy = re^{-i\theta}$. $|z| = \sqrt{x^2 + y^2} \Rightarrow |z_1 z_2| = |z_1||z_2|$, $|z_1 + z_2| \leq |z_1| + |z_2|$

Cauchy-Riemann Equations:

($\frac{d}{dz}(\bar{z})$ does not exist) $f(x,y) = u(x,y) + iv(x,y) \Rightarrow u_x = v_y$, $u_y = -v_x \Leftrightarrow f$ is differentiable. If f is smooth on a disk, it is analytic.

A curve is a differentiable function $\gamma : [a,b] \to \mathbb{C}$ with $\gamma'(t) \neq 0$ that doesn't intersect itself. Curves to note:



$\gamma_1(t) = a + e^{it}$, $t \in [0, 2\pi]$

$\gamma_2(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$

A contour is a concatenation of a finite num. of simple curves meeting at endpoints. If it does not intersect itself, it is simple. A closed contour ends where it starts.

A simple closed contour is positively oriented if the interior (the bounded region) is to the left of the direction of motion.

Cauchy-Goursat Theorem: If $f(z)$ is analytic on & inside a simple closed contour C, then $\oint_C f(z)dz = 0$.

$\Rightarrow$ Independence of Path: If f is analytic on a simply connected region R & $z_1, z_2 \in R$, then

$\log(z) = \log|z| + i\theta + 2\pi i k$, $k \in \mathbb{Z}$. $Log(z) = \log|z| + Arg(z)$. $Arg(re^{i\theta}) = \phi$, where $e^{i\phi} = e^{i\theta}$, but $\phi \in [0, 2\pi)$

$a^b = e^{b \log a}$; $\cos(z) = (e^{iz} + e^{-iz})/2$, $\sin(z) = (e^{iz} - e^{-iz})/2i$; A Region $R \subseteq \mathbb{C}$ is an open subset of $\mathbb{C}$ that is connected.

$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$  for any C_1, C_2 from z_1 to z_2

$\Rightarrow$ Cauchy's Integral formula: If f is analytic on & inside a simple closed positively oriented contour C, then for any $a \in$ interior, $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \Leftrightarrow f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$

$\Rightarrow$ Infinite Differentiability: If f is analytic at z, then its differentiable in a disk containing z.

$f^{(n)}(z) = \frac{1}{2\pi i} n! \oint \frac{f(w)}{(w-z)^{n+1}} dw$

A point where $f(z)$ is not analytic is called a singularity. If it's analytic in a punctured disk around the pt, its an isolated sing

Laurent Series: A series that decomposes a function into z^n & $\frac{1}{z^n}$ terms. Converges on an annulus.

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$. The coeff. b_1 is known as the residue of $f(z)$.

Note: Annuli of convergence end at singularities. Residue Thm: If f has isolated singularities at $z_0, z_1, \dots, z_n$, where $\{z_i\} \subseteq R$ and C bounds R, $\oint_C f(z)dz = 2\pi i \left(\sum_i^n Res(z_i)\right)$.

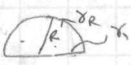
Methods for Finding Residues Quickly: ① Simple Pole: $f(z) = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \Leftrightarrow \lim_{z \to z_0} (z - z_0) f(z) = b_1$. ② Higher order poles:

$Res(z_0) = \frac{1}{m-1} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$.

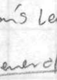
Applications of Residue Theorem

$\int_0^{2\pi} \frac{P(\cos\theta, \sin\theta)}{Q(\cos\theta, \sin\theta)} d\theta$, $Q \neq 0$. $\Rightarrow$ use $\gamma(t) = e^{it} \to \cos\theta = \frac{z + z^{-1}}{2}$, $\sin\theta = \frac{z - z^{-1}}{2i}$ $\Rightarrow$ Use Residue Theorem.

$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$, $Q \neq 0 \Rightarrow$ use $\gamma_1 + \gamma_R(t) = Re^{it}$, $t \in [0, \pi]$



Jordan's Lemma: $\lim_{R \to \infty} \int_{\gamma_R} \frac{P(z) e^{imz}}{Q(z)} dz = 0$; $\gamma_R(t) = Re^{it}$, $t \in [0, \pi]$ ($\deg(P) \leq \deg(Q) - 1$

In general:  $\gamma = re^{it}$, $t \in [\theta_1, \theta_2] \Rightarrow \int_\gamma f(z)dz = i\,Res(z_0) \cdot (\theta_2 - \theta_1)$