How to Do Things with Differential Equations and Eigenvalues and Stuff

Diagonalization – finding the diagonal matrix and the P that goes with it:

1. Compute eigenvalues of A.
2. Find corresponding eigenvectors
3. Construct P and D:
    a. P = [$\mathbf{v}_1$ $\mathbf{v}_2$] where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors from Step 2.
    b. D = a diagonal matrix with the corresponding eigenvalues.
4. Check if AP = PD! **State this**. You might not have to do PDP^{-1} and save yourself some time.
5. Check if P is invertible.
6. Compute P^{-1}!
    a. <u>Necessary step if asked to find A^{-1}, because</u> $\underline{\mathbf{A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}}}$

Verify linear independence of a set of vector functions:

1. Check that the Wronskian is nonzero!

Verify that a set is a fundamental solution for a system:

1. Find the 1st derivative of each set.
2. Plug the 1st derivative and original **x** vector from the set into the equation.
3. Check that the Wronskian of all vectors in the set is nonzero.

Find a General Solution of a System x'(t) = Ax(t)

1. Find eigenvalues and eigenvectors of A.
2. If there are no multiplicities, form one solution per eigenvector using the form $\mathbf{x}_n = c_n e^{rt}\mathbf{u}_n$. Skip to step 4.
3. If there are multiplicities, it's time for some fun!
    a. Use Lu's method of solving for each generalized eigenvector using the previous one and A-rI (starting from the initial one found from A-rI) repeatedly, forming a chain using the equation $e^{rt}(\mathbf{v}_k + \mathbf{v}_{(k-1)}t + \mathbf{v}_{(k-2)}t^2/2!\ ...)$. Each chain you form forms a separate solution, so you will have *k* solutions for this repeated eigenvalue with anywhere from 1 to *k* terms.
        i. Notice how this is similar to the power series expansion, except the power series expansion for each solution involved the matrix $(A-rI)^k$ multiplied by the **same** eigenvector **u** (which was actually the last generalized eigenvector $\mathbf{v}_k$ you found using this method).
        ii. <u>*Remember*</u>: for each subsequent solution, the 'older' the generalized eigenvector, the higher the power it is raised to!

This illustrates it fairly well:

<u>Regardless of which method you use: GENERALIZED EIGENVECTORS DO NOT BELONG UNDER ANY CIRCUMSTANCES IN A FUNDAMENTAL MATRIX. You must find the corresponding regular eigenvector!</u>

Such vectors are called *generalized eigenvectors*. For every eigenvector $\vec{v}_1$ we find a chain of generalized eigenvectors $\vec{v}_2$ through $\vec{v}_k$ such that:

$$(A - \lambda I)\vec{v}_1 = \vec{0},$$
$$(A - \lambda I)\vec{v}_2 = \vec{v}_1,$$
$$\vdots$$
$$(A - \lambda I)\vec{v}_k = \vec{v}_{k-1}.$$

We form the linearly independent solutions

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t},$$
$$\vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t)e^{\lambda t},$$
$$\vdots$$
$$\vec{x}_k = \left(\vec{v}_k + \vec{v}_{k-1}t + \vec{v}_{k-2}\frac{t^2}{2} + \cdots + \vec{v}_2\frac{t^{k-2}}{(k-2)!} + \vec{v}_1\frac{t^{k-1}}{(k-1)!}\right)e^{\lambda t}.$$

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$ is the factorial. We proceed to find chains until we form m linearly independent solutions (m is the multiplicity). You may need to find several chains for every eigenvalue.

4. Sum up each solution and you have the general solution to the homogeneous system with constant coefficients!
    a. <u>*Shortcut*</u>: if multiplicity = 2, the general solution, after finding $\mathbf{v_2}$ from (A-rI) and $\mathbf{v_3}$ from $(A\text{-}rI)^2$ will be of the form $e^{rt}(t\mathbf{v_2} + \mathbf{v_3})$
5. If required, form the fundamental matrix $X(t) = [e^{rt}\mathbf{u}_1 \cdots e^{rt}\mathbf{u}_n]$
6. If given initial conditions, plug t_0 into the fundamental matrix and row reduce (or, if it's simple enough, write out a system of equations from) the augmented matrix $[X(t_0) \mid \mathbf{x_0}]$ to solve for **c.**
    a. If finding $e^{\mathbf{A}t}$ is part of the problem, just multiply $e^{\mathbf{A}t}$ by the vector $\mathbf{x}_0$ to find the required solution.
7. If asked to find $e^{\mathbf{A}t}$**,** calculate X(0), invert it to find $X(0)^{-1}$, and solve:
    $e^{\mathbf{A}t} = X(t)\,X(0)^{-1}$

Dealing with Complex Eigenvalues

1. Once you've found an eigenvalue *r* is complex, row reduce the matrix A-*r*I as usual.
2. When you've found your eigenvector, decompose it into the real and imaginary parts, with the *i* pulled out of the imaginary vector.
    a. These two vectors form two linearly independent solutions. Let's call the one associated with the real portion **a** and the imaginary portion **b**
3. The two linearly independent solutions that form one <u>*general solution*</u> are:
    a. $e^{\alpha t}\cos\beta t\,\mathbf{a} - e^{\alpha t}\sin\beta t\,\mathbf{b}$
    b. $e^{\alpha t}\sin\beta t\,\mathbf{a} + e^{\alpha t}\cos\beta t\,\mathbf{b}$

Method of Undetermined Coefficients

1. Solve for homogenous equation (see above)
2. Make a guess for $\mathbf{x}_p$ using f(t) as a guideline.
    a. Use the principle of superposition if necessary, introducing as many vectors of variables as necessary for each term in f(t)
    b. Remember to add a factor of *t* if *e* is raised to an eigenvalue of the homogenous solution!
3. Find $\mathbf{x}'_p$ and plug the particular guess and its derivative into the system.
4. Equate coefficients.

5. Solve for the system with the fewest numbers of unknowns first using an augmented matrix, then solve for the others.
    a. Remember that if you equated things to zero initially, the right side of the augmented matrix will be negative! (e.g.: 0 = A**a**t + **g**t → -**g**=A**a**)
    b. Once you solve one system you can solve them all just by performing the same row operations on the right side of the augmented matrix!
6. Once you solve, plug coefficients back in to find particular solution!

Variation of Parameters (With Boundary Conditions)

1. Solve for homogenous equation (see above)
2. Form fundamental matrix.
3. Find its inverse.
4. Acknowledge that since $X(t_0)\mathbf{c} = \mathbf{x}_0$, $\mathbf{c} = X^{-1}(t_0)\mathbf{x}_0$
5. Plug into this equation:

Solving for $\mathbf{c}$, we have $\mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0$. Thus, the solution to (12) is given by the formula

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \;. \tag{13}$$

$\vec{x}_h = c_1 e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-1\\2\end{bmatrix}$

Now find $\vec{x}_p$: form $= t\vec{a} + \vec{b} = \vec{x}_p$ $\quad \vec{x}_p' = \vec{a}$

$\vec{x}' = A\vec{x} + f(t)$ plug in $\vec{x}_p$ & $\vec{x}_p'$

$\vec{a} = A(\vec{a}t + \vec{b}) + \vec{g}t + \vec{h}$ ($\vec{g}$ and $\vec{h}$ defined at top of page)

$\vec{a} = A\vec{a}t + A\vec{b} + \vec{g}t + \vec{h}$

$A\vec{a}t + A\vec{b} + \vec{g}t + \vec{h} - \vec{a} = 0$

Collect like terms: $(A\vec{a} + \vec{g})t + (A\vec{b} + \vec{h} - \vec{a}) = 0$

then set coefficients equal to zero

$A\vec{a} + \vec{g} = 0 \rightarrow A\vec{a} = -\vec{g}$ row reduce to solve for $\vec{a}$:

$\left[\begin{array}{cc|c}1&1&1\\4&1&4\end{array}\right] \underset{R2\to R2-4R1}{\sim} \left[\begin{array}{cc|c}1&1&1\\0&-3&0\end{array}\right] \underset{R2\to -\frac{1}{3}R2}{\sim} \left[\begin{array}{cc|c}1&1&1\\0&1&0\end{array}\right] \underset{R1\to R1-R2}{\sim} \left[\begin{array}{cc|c}1&0&1\\0&1&0\end{array}\right]$ so $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$

now $A\vec{b} = \vec{a} - \vec{h} = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

$\left[\begin{array}{cc|c}1&1&2\\4&1&2\end{array}\right] \underset{R2\to R2-4R1}{\sim} \left[\begin{array}{cc|c}1&1&2\\0&-3&-6\end{array}\right] \underset{R2\to -\frac{1}{3}R2}{\sim} \left[\begin{array}{cc|c}1&1&2\\0&1&2\end{array}\right]$ $\quad x_2 = 2 \quad x_1 + x_2 = 2 \quad x_1 = 0$

$\begin{bmatrix}0\\2\end{bmatrix}$

then $\vec{x}_p = \vec{a}t + \vec{b} = \begin{bmatrix}1\\0\end{bmatrix}t + \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}t\\2\end{bmatrix}$

Relevant Theorems:

THEOREM 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad \text{where} \quad P = [\,\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}\,] \quad \text{and} \quad C = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$$

1.

a. **Know these too:**

25. Let A be a real $n \times n$ matrix, and let $\mathbf{x}$ be a vector in $\mathbb{C}^n$. Show that $\operatorname{Re}(A\mathbf{x}) = A(\operatorname{Re}\mathbf{x})$ and $\operatorname{Im}(A\mathbf{x}) = A(\operatorname{Im}\mathbf{x})$.

26. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$.

a. Show that $A(\operatorname{Re}\mathbf{v}) = a\operatorname{Re}\mathbf{v} + b\operatorname{Im}\mathbf{v}$ and $A(\operatorname{Im}\mathbf{v}) = -b\operatorname{Re}\mathbf{v} + a\operatorname{Im}\mathbf{v}$. [*Hint:* Write $\mathbf{v} = \operatorname{Re}\mathbf{v} + i\operatorname{Im}\mathbf{v}$, and compute $A\mathbf{v}$.]

b. Verify that if P and C are given as in Theorem 9, then $AP = PC$.

2. **Derivation between (7) and (8)**

Generalized Eigenvectors

Definition 5. Let $\mathbf{A}$ be a square matrix. A nonzero vector $\mathbf{u}$ satisfying

$$(\mathbf{A} - r\mathbf{I})^m \mathbf{u} = \mathbf{0} \tag{7}$$

for some scalar r and some positive integer m is called a **generalized eigenvector** associated with r.

[Note that r must be an eigenvalue of $\mathbf{A}$, since the final *nonzero* vector in the list $\mathbf{u}$, $(\mathbf{A} - r\mathbf{I})\mathbf{u}$, $(\mathbf{A} - r\mathbf{I})^2\mathbf{u}, \ldots, (\mathbf{A} - r\mathbf{I})^{m-1}\mathbf{u}$ is a "regular" eigenvector.]

A valuable feature of generalized eigenvectors $\mathbf{u}$ is that we can compute $e^{\mathbf{A}t}\mathbf{u}$ in finite terms *without knowing* $e^{\mathbf{A}t}$, because

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{u} &= e^{r\mathbf{I}t}e^{(\mathbf{A}-r\mathbf{I})t}\mathbf{u} \\ &= e^{rt}\left[\mathbf{I}\mathbf{u} + t(\mathbf{A} - r\mathbf{I})\mathbf{u} + \cdots + \frac{t^{m-1}}{(m-1)!}(\mathbf{A} - r\mathbf{I})^{m-1}\mathbf{u} + \frac{t^m}{m!}(\mathbf{A} - r\mathbf{I})^m\mathbf{u} + \cdots\right] \\ &= e^{rt}\left[\mathbf{u} + t(\mathbf{A} - r\mathbf{I})\mathbf{u} + \cdots + \frac{t^{m-1}}{(m-1)!}(\mathbf{A} - r\mathbf{I})^{m-1}\mathbf{u} + \quad \mathbf{0} \quad + \cdots\right]. \end{aligned} \tag{8}$$

a.