ME 109

Bradley Cage

1-D, steady state sol to heat eq. $\dot{q} \neq 0$

	Plane Wall	Cylindrical Wall	Sphere Wall
Heat eq.	$\frac{d^2T}{dx^2}=0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$
Temp Dist.	$T_{s,1}-\Delta T\frac{x}{L}$	$T_{s,2}+\Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1}-\Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
q''	$k\frac{\Delta T}{L}$	$\frac{k\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
q	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\Delta T}{(1/r_1)-(1/r_2)}$
$R_{t,cond}$	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

Boundary Conditions

- Constant $T_s \rightarrow T(0,t)=T_s$
- Constant q'' $\rightarrow$ Finite flux $\rightarrow q''(x=0) = -k\frac{\partial T}{\partial x}\Big|_{x=0} = q''_s$
  - $\rightarrow$ Adiabat/Insulated $\rightarrow \frac{\partial T}{\partial x}\Big|_{x=0}=0$
- Convective surface.
  - $\rightarrow -k\frac{\partial T}{\partial x}\Big|_{x=0} = h[T_\infty - T(0,t)]$

Thermal Resistance

- $R_{t,cond} = \frac{T_{s,1}-T_{s,2}}{q_x} = \frac{L}{kA}$
- $R_{t,conv} = \frac{T_s - T_\infty}{q} = \frac{1}{hA}$
- $R_{t,rad} = \frac{T_{surr}-T_{sur}}{q_{rad}} = \frac{1}{h_rA}$
- Can write q as sum

i.e. $\left\{ q_x = \frac{T_{\infty,1}-T_{\infty,2}}{\frac{1}{h_1A}+\frac{L}{kA}+\frac{1}{h_2A}} \right\}$ plane wall w/ convection.

- Draw resistance diagrams.

$q_x \longrightarrow T_{\infty,1} \overset{\frac{1}{h_1A}}{\text{---}} T_{s,1} \overset{\frac{L}{kA}}{\text{---}} T_{s,2} \overset{\frac{1}{h_2A}}{\text{---}} T_{\infty,2}$

Fourier's Law

- conduction $\rightarrow q''_x = -k\frac{dT}{dx}$ (basic)
- convection $\rightarrow q'' = h(T_s - T_\infty)$
- Vectorial
  - $q'' = -k\nabla T \Rightarrow q''_x = -k\frac{\partial T}{\partial x}$, $q''_y = -k\frac{\partial T}{\partial y}$, etc...
- Heat rate
  - $q_x = -kA\frac{dT}{dx} \Rightarrow q''_x = \frac{q_x}{A}$

Heat Diffusion

- General $\longrightarrow q_{x+dx} = q_x + \frac{\partial q_x}{\partial x}dx$, etc.
- Based on $\dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out} = \dot{E}_{st}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q} = \underbrace{\rho c_p\frac{\partial T}{\partial t}}_{\dot{E}_{st}}$$

- $\dot{E}_{st}$ normally $=0$.
- For one-D (w/ heat gen) $\rightarrow \frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{q} = 0$
- Thermal Diffusivity
  - $\hookrightarrow \alpha = k/\rho c_p$

Fins + Extended Surfaces

- General Form of Energy eq.

$$\frac{d^2T}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx}-\left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T-T_\infty)=0$$

- with $\theta = T - T_\infty$

$$\frac{d^2\theta}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{d\theta}{dx}-\left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)\theta=0$$

- For uniform cross-section

$$\frac{d^2T}{dx^2}-\frac{hP}{kA_c}(T-T_\infty)=0 \qquad P = \text{perimeter}$$

- More commonly

$$\frac{d^2\theta}{dx^2}-m^2\theta=0 \quad \text{where } m^2=\frac{hP}{kA_c}$$

- Sol $\rightarrow \theta(x) = C_1e^{mx}+C_2e^{-mx}$

Temp. dist. and heat loss for fins of c^{st} cross sec

case	Tip Condition	Temp. Dist θ/θ_b	Heat rate q_f
A	Convective heat transfer $h\theta(L) = -k\,d\theta/dx\mid_{x=L}$	$\frac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$	$M\frac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+(h/mk)\sinh mL}$
B	Adiabatic $d\theta/dx\mid_{x=L}=0$	$\frac{\cosh m(L-x)}{\cosh mL}$	$M\tanh(mL)$
C	Prescribed T. $\theta(L)=\theta_L$	$\frac{(\theta_L/\theta_b)\sinh mx+\sinh m(L-x)}{\sinh mL}$	$M\frac{(\cosh mL-\frac{\theta_L}{\theta_b})}{\sinh mL}$
D	∞ fin $\theta(L)=0$	e^{-mx}	M

$\theta = T - T_\infty \qquad m^2 = hP/kA \qquad \theta_b = \theta(0) = T_b - T_\infty \qquad M = \sqrt{hPkA_c}\,\theta_b$

Fin Effectiveness

- $\varepsilon_f = q_f/(hA_{c,b}\theta_b)$ where $A_{c,b}$ = fin cross sec. at base.
- Resistance $\rightarrow R_{t,f} = \theta_b/q_f$
- R at base $\rightarrow R_{t,b} = 1/hA_{c,b}$

$\left.\right\} \varepsilon_f = \frac{R_{t,b}}{R_{t,f}}$

- Max fin efficiency $\rightarrow \eta_f = q_f/q_{max} = q_f/(hA_f\theta_b)$

Straight Fins

- Rectangular
  - $A_f = 2wL_c$
  - $L_c = L + (t/2)$
  - $A_p = tL$
  - $\eta_f = \frac{\tanh mL_c}{mL_c}$
- Triangular
  - $A_f = 2w(L^2+(t/2)^2)^{1/2}$
  - $A_p = (t/2)L$
  - $\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$

Shape Factor

- $q = Sk\Delta T_{1-2}$
- $R_{t,cond} = \frac{1}{Sk}$

Dimensionless Heat Cond. Rate

- L_c for ∞ medium cases
  - $L_c = \sqrt{\frac{A_s}{4\pi}}$
- $q^*_{ss} = \frac{qL_c}{kA_s(T_1-T_2)}$