Wohl Ch. 3: Simple Approximations

1. Dimensions & Scaling
2. Fitting Wavelengths in a Well: WKB
3. Guessing the Ground State: Variational Method

Dimensions & Scaling

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$ All 3 terms have to have same dimensions

Energies of hydrogen atom $\left(E_n=-\frac{1}{n^2}\frac{\hbar^2}{2ma_0^2}\right)$ & infinite square well $\left(E_n=n^2\frac{\pi^2\hbar^2}{2mL^2}\right)$

$a_0=\frac{\hbar^2}{mke^2}$ is the Bohr radius & L is the width of the square well.

Schrödinger for harmonic oscillator: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$

Let M, L, T represent the fundamental physical dimensions of mass, length, time.

$\dim(E)=ML^2T^{-2}$ $\dim(m)=M$ $\dim(\omega)=T^{-1}$ $\dim(\hbar)=ML^2T^{-1}$

$\dim(ke^2)=ML^3T^{-2}$

Ex: $\dim(E)=[\dim(\hbar)]^a[\dim(m)]^b[\dim(ke^2)]^c$

$\Rightarrow ML^2T^{-2}=(ML^2T^{-1})^aM^b(ML^3T^{-2})^c$

$1=a+b+c$ $2=2a+3c$ $-2=-a-2c$ solve. $E\propto$ result

Fitting Wavelengths in a Well: The WKB Method

- For many potential wells of interest, no solutions to the Schrödinger eqns. can be solved in closed form

WKB method:

1) Calculate # of de Broglie wavelengths between turning points of a well as a function of an arbitrary value of the energy.
2) How many wavelengths correspond to the quantized energy levels of the infinite square well & the oscillator, where we already know the answers?

Ex: infinite square well

$$\lambda=\frac{h}{p}=\frac{2\pi\hbar}{\sqrt{2mE}} \quad \left(E=\frac{p^2}{2m}\right)$$

wavelengths between 0 & L $=\int_0^L\frac{dx}{\lambda}=\frac{L}{\lambda}=\frac{\sqrt{2mE}}{2\pi\hbar}L$

quantized energy levels: # wavelengths $=\frac{n}{2}$

$$\Rightarrow E_n=n^2\frac{\pi^2\hbar^2}{2mL^2}$$

Ex: A bouncing ball: $V(z)=\begin{cases} mgz & z\geq 0 \\ \infty & z<0 \end{cases}$ $\quad * \ \lambda(z)=\frac{h}{p(z)}=\frac{2\pi\hbar}{\sqrt{2m(E-V(z))}}$ $\left(\frac{p^2}{2m}=E-V(z)\right)$

$\Rightarrow$ Turning points at $z=0$, $z=\frac{E}{mg}\equiv z_0$

$$\int_0^{z_0}\frac{dz}{\lambda(z)}=\frac{1}{2\pi\hbar}\int_0^{z_0}\sqrt{2m(E-mgz)}\,dz$$

$y\equiv z/z_0$

$$\Rightarrow \frac{\sqrt{2mE}}{2\pi\hbar}\frac{E}{mg}\int_0^1\sqrt{1-y}\,dy=\frac{1}{2\pi g\hbar}\sqrt{\frac{2E^3}{m}}\left(-\frac{2}{3}\right)(1-y)^{3/2}\Big|_0^1=\frac{1}{3\pi g\hbar}\sqrt{\frac{2E^3}{m}}$$

want $(n/2-1/8)$ wavelengths $\Rightarrow \frac{1}{3\pi gh}\sqrt{\frac{2E_n^3}{m}}\approx\frac{1}{2}\left(n-\frac{1}{4}\right)$

offset chosen by: $\begin{cases} \text{2 hard turning points: } n=0,1,2;\ \frac{1}{4},\frac{3}{4},\frac{5}{4} \\ \text{1 soft, 1 hard: } n=1,2,3;\ \frac{3}{8},\frac{7}{8},\frac{11}{8} \\ \text{2 soft; } n=0,1,2\ldots;\ 1/4,\ 3/4,\ 5/4 \end{cases}$

Guessing the Ground State: The Variational Method

- Used mainly to get a good approximation for the ground-state energy of a system.
- If ψ_1 is the true gnd. state ~~energy~~ wavefunction & $\hat{H}$ is the Hamiltonian, then $\hat{H}\psi_1=E_1\psi_1$

$$\int_{-\infty}^{\infty}\psi_1^*(\hat{H}\psi_1)dx=E_1\int_{-\infty}^{\infty}\psi_1^*\psi_1\,dx=E_1$$

- We don't know ψ_1 so we guess a trial wavefunction ψ_T.

$$E_T=\int_{-\infty}^{\infty}\psi_T^*(\hat{H}\psi_T)dx$$

$\psi_T=\sum_{n=1}^{\infty}c_n\psi_n$ where $\sum|c_n|^2=1$ $\qquad E_T=\sum_{n=1}^{\infty}E_n|c_n|^2\geq\sum_{n=1}^{\infty}E_1|c_n|^2=E_1\rightarrow E_T\geq E_1$

Ex) hydrogen: $\psi_T(r)=ce^{-br}$ (true gnd state wavefunction has $b=1/a_0$)

$*\ \int_0^{\infty}r^ne^{-ar}dr=\frac{n!}{a^{n+1}}$, $\quad \int_{\text{space}}|\psi_T(r)|^2r^2\sin\theta\,dr\,d\theta\,d\phi=4\pi|c|^2\int_0^{\infty}e^{-2br}r^2dr$

$$=4\pi|c|^2\frac{2}{(2b)^3}=\frac{\pi|c|^2}{b^3}=1$$

$V(r)=-ke^2/r$ $\rightarrow$ potential energy part of integral of $\psi_T(\hat{H}\psi_T)=-4\pi ke^2|c|^2\int_0^{\infty}e^{-2br}r\,dr=-ke^2b$

Kinetic energy integral: $\nabla\psi_T=c\frac{d(e^{-br})}{dr}\hat{r}=-b\psi_T\hat{r}$

$$-\frac{\hbar^2}{2m}\int_{\text{space}}\psi_T^*\nabla^2\psi_T\,d^3\underline{r}=+\frac{\hbar^2}{2m}b^2\int_{\text{space}}|\psi_T|^2d^3\underline{r}=\frac{\hbar^2b^2}{2m}$$

$$E_T(b)=\frac{\hbar^2b^2}{2m}-ke^2b$$

$$\frac{dE_T}{db}=\frac{\hbar^2b}{m}-ke^2=0$$

$$\Rightarrow b=\frac{mke^2}{\hbar^2}=\frac{1}{a_0}$$

Wohl Ch. 10: Spin-1/2 Particles

1. Spinors: Eigenvals & Eigenstates
2. Polarization Vector
3. Magnetic Moments & Magnetic Fields
4. Time Dependence: Precessing the Polarization
5. Time Dependence: Flipping the Polarization
6. Stern-Gerlach Experiments

- In a magnetic field, the spin-up & down states w.r.t. the field direction have different energies (Zeeman Effect)
- In a static magnetic field, the spin precesses about the field direction (Larmor precession)
- In a properly arranged time-dependent field, the spin will spiral back & forth between spin-up & down states (magnetic resonance)
- In an inhomogeneous field, the spin-up & down states can be separated into two beams (Stern-Gerlach)

Spinors

- The only simultaneous eigenstates of the angular momentum operators $\hat{S}^2$ & $\hat{S}_z$ are:

$|s,m\rangle = |1/2, \pm 1/2\rangle$

shorthand: $|+z\rangle = |+\rangle = |\alpha\rangle = |1\rangle = \uparrow$

$|-z\rangle = |-\rangle = |\beta\rangle = |2\rangle = \downarrow$

*In spin up, $|+z\rangle$, $\angle \underline{s} = 55°$ w.r.t. $+z$

Two component vectors (spinors): $|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|-z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $|+x\rangle = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ $|-x\rangle = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$

$|\chi\rangle = c_+|+z\rangle + c_-|-z\rangle$ $|+y\rangle = \begin{pmatrix}1/\sqrt{2}\\i/\sqrt{2}\end{pmatrix}$ $|-y\rangle = \begin{pmatrix}1/\sqrt{2}\\-i/\sqrt{2}\end{pmatrix}$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad [\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

General Eigenstates

$a\hat{I} + \vec{b}\cdot\hat{\vec{\sigma}}$ eigenstates belonging to eigenvals $a \pm b$ are associated w/ $\underline{b} = b\underline{\hat{n}}$

$(b_x, b_y, b_z) = (b\sin\theta\cos\phi, b\sin\theta\sin\phi, b\cos\theta)$

$$a\hat{I} + \vec{b}\cdot\vec{\sigma} = \begin{pmatrix}a+b\cos\theta & be^{-i\phi}\sin\theta\\ be^{+i\phi}\sin\theta & a-b\cos\theta\end{pmatrix}$$

eigenvals $\lambda_\pm = a \pm b$ → eigenstates $|+n\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$, $|-n\rangle = \begin{pmatrix}\sin\frac{\theta}{2}\\ -e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}$

Polarization Vector

$\underline{P}$ points along direction which, were a measurement made of the component of $\underline{s}$, the result would be $+\hbar/2$ w/ certainty. $\underline{P}$ points in direction along which spin is up.

$(P_x, P_y, P_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $P_x^2 + P_y^2 + P_z^2 = 1$

$\underline{P} \overset{\text{(one to one)}}{=} |+n\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$ (spin up/down amplitudes) or write in x,y,z / r,θ,ϕ

$c_+ = \langle +z|\chi\rangle$ $c_- = \langle -z|\chi\rangle$

$P_x = 2Re(c_+^* c_-)$ $P_y = 2Im(c_+^* c_-)$ $P_z = |c_+|^2 - |c_-|^2$

$$\underline{P} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} = \begin{pmatrix}2Re(c_+^* c_-)\\ 2Im(c_+^* c_-)\\ |c_+|^2 - |c_-|^2\end{pmatrix}$$

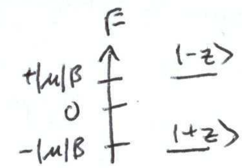
- Suppose, knowing an initial $\underline{P}$, want the probability of getting $+\hbar/2$ on measuring along the direction of arbitrary $\underline{d}$.

$$P(\text{spin up along } \underline{d}) = \frac{1+\cos\beta}{2} = \cos^2\left(\frac{\beta}{2}\right)$$ where β is angle between $\underline{P}$ & $\underline{d}$.

Magnetic Moments & Magnetic fields

$\mu = IA = qfA = \frac{qv}{2\pi r}\pi r^2 = \frac{q}{2m}L$ for flat current loop. $L = mvr$

$\Rightarrow \boxed{\underline{\mu} = \frac{q}{2m}\underline{L}}$ For spin-1/2 $\underline{\mu} = g\frac{q}{2m}\underline{S} = g\frac{q\hbar}{4m}\underline{\sigma} = \mu\underline{\sigma}$



Electron: $\underline{\mu}_e = -\mu_B\underline{\sigma}$ where $\mu_B = e\hbar/2m$

Interaction of magnetic moment w/ magnetic field

$$V = -\underline{\mu}\cdot\underline{B} \to \hat{H} = -\underline{\mu}\cdot\underline{B} = -\mu\hat{\underline{\sigma}}\cdot\underline{B} = -\mu(\hat{\sigma}_x B_x + \hat{\sigma}_y B_y + \hat{\sigma}_z B_z) = -\mu\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Time Dependence: Precessing the Polarization

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi$$

$$|\chi(t)\rangle = \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix} = \begin{pmatrix} c_+(0)e^{+i\omega_0 t/2} \\ c_-(0)e^{-i\omega_0 t/2} \end{pmatrix}$$

$P_x(t) = 2\,\mathrm{Re}[c_+^*(t)c_-(t)] = 2\,\mathrm{Re}[c_+^*(0)c_-(0)e^{-i\omega_0 t}]$

$P_y(t) = 2\,\mathrm{Im}[c_+^*(t)c_-(t)] = 2\,\mathrm{Im}[c_+^*(0)c_-(0)e^{-i\omega_0 t}]$

$P_z(t) = \cos\theta$

say $\underline{P}$ at $t_0 = 0$ to lie in xz plane $\quad \hbar\omega_0 \equiv 2\mu B_0$

$P_x(t) = \sin\theta\cos\omega_0 t \quad P_y(t) = -\sin\theta\sin\omega_0 t, \quad P_z(t) = P_z(0) = \cos\theta$

Time Dependence: Flipping the Polarization

add to the static $\underline{B}_0$ a field $\underline{B}_1$ that rotates in the xy plane.

$$\underline{B} = \underline{B}_0 + \underline{B}_1 = B_0\hat{z} + B_1(\hat{x}\cos\omega t - \hat{y}\sin\omega t)$$

$$\hat{H} = -\underline{\mu}\cdot(\underline{B}_0 + \underline{B}_1) = -\mu B_0\hat{\sigma}_z - \mu B_1(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t) = -\mu\begin{pmatrix} B_0 & B_1e^{+i\omega t} \\ B_1e^{-i\omega t} & -B_0 \end{pmatrix}$$

Schrödinger: $i\hbar\begin{pmatrix} \dot{c}_+ \\ \dot{c}_- \end{pmatrix} = -\mu\begin{pmatrix} B_0 & B_1e^{+i\omega t} \\ B_1e^{-i\omega t} & -B_0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \quad \to \omega_0 \equiv 2\mu B_0/\hbar$

$\omega_1 = 2\mu B_1/\hbar$

$$\Rightarrow \frac{dc_+}{dt} = \frac{i\mu}{\hbar}(B_0c_+ + B_1e^{+i\omega t}c_-) = \frac{i}{2}(\omega_0c_+ + \omega_1e^{+i\omega t}c_-)$$

$$\frac{dc_-}{dt} = \frac{i\mu}{\hbar}(B_1e^{-i\omega t}c_+ + B_0c_-) = \frac{i}{2}(\omega_1e^{-i\omega t}c_+ - \omega_0c_-)$$

uncoupling:

$$\Rightarrow c_+(t) = a_+(t)e^{+i\omega_0 t/2} = e^{i\omega t/2}(Ce^{+i\Omega t/2} + De^{-i\Omega t/2})$$

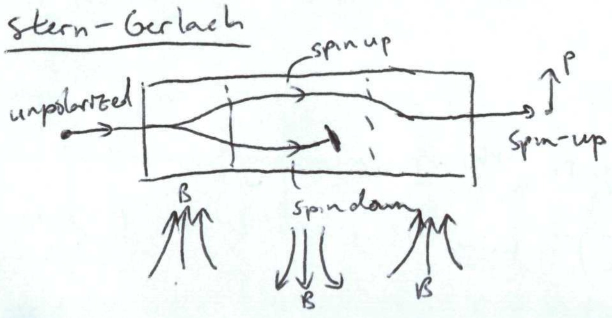
$$c_-(t) = e^{-i\omega t/2}(Fe^{+i\Omega t/2} + Ge^{-\Omega t/2})$$

$|c_-(t)|^2 = \frac{\omega_1^2}{\Omega^2}\sin^2(\frac{1}{2}\Omega t) = A(\omega)\sin^2(\frac{\Omega}{2}t)$ where $\Omega^2 \equiv (\omega-\omega_0)^2 + \omega_1^2$

$A(\omega) = \frac{\omega_1^2}{\Omega^2} = \frac{1}{1 + (\omega-\omega_0)^2/\omega_1^2}$

*The function $A(\omega)$ resonates at $\omega = \omega_0$

Stern-Gerlach



Wohl Ch.11: Two Angular Momenta

1. Hyperfine Structure of the Hydrogen Ground State
2. The 21-cm Line & Astronomy
3. Total Spin of Two Spin-1/2 Particles
4. Coupling Any Two Angular Momenta
5. Clebsch-Gordan Coefficients
6. Particle Multiplets & Isospin

Hyperfine

Two spin-1/2 particles

basis states: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

Ex) $|+-\rangle \to |s_e, s_p; m_e, m_p\rangle = |\tfrac{1}{2}, \tfrac{1}{2}; +\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$$|++\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |+-\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |-+\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |--\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Gnd st. of Hydrogen

$$\Rightarrow \psi_{100}(r)|\chi_0\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \begin{pmatrix}c_{++}\\c_{+-}\\c_{-+}\\c_{--}\end{pmatrix}$$

Two magnetic moments associated w/ the spins of electron & proton

$\hat{\mu}_e = -\mu_B \hat{\sigma}_e \quad \hat{\mu}_p = \mu_p \hat{\sigma}_p \quad E_{hydrogen} = -\frac{13.6\,eV}{n^2}$

$$\begin{pmatrix}a_1\\b_1\end{pmatrix} \otimes \begin{pmatrix}a_2\\b_2\end{pmatrix} = \begin{pmatrix}a_1 b_1\\a_1 b_2\\b_1 a_2\\b_1 b_2\end{pmatrix}$$

$$\hat{H}_{hf} = A\hat{\sigma}_e \cdot \hat{\sigma}_p = A(\hat{\sigma}_x^e \hat{\sigma}_x^p + \hat{\sigma}_y^e \hat{\sigma}_y^p + \hat{\sigma}_z^e \hat{\sigma}_z^p)$$

$$\begin{pmatrix}a_1 & c_1\\b_1 & d_1\end{pmatrix} \otimes \begin{pmatrix}a_2 & c_2\\b_2 & d_2\end{pmatrix} = \begin{pmatrix}a_1a_2 & a_1c_2 & c_1a_2 & c_1c_2\\a_1b_2 & a_1d_2 & c_1b_2 & c_1d_2\\b_1a_2 & b_1c_2 & d_1a_2 & d_1c_2\\b_1b_2 & b_1d_2 & d_1b_2 & d_1d_2\end{pmatrix}$$

$\hat{\sigma}_x|+\rangle = +|-\rangle \quad \hat{\sigma}_y|+\rangle = +i|-\rangle \quad \hat{\sigma}_z|+\rangle = +|+\rangle$

$\hat{\sigma}_x|-\rangle = +|+\rangle \quad \hat{\sigma}_y|-\rangle = -i|+\rangle \quad \hat{\sigma}_z|-\rangle = -|-\rangle$

21-cm line

splits to $+A$ and $-3A$

$\nu = 4A/h, \quad \lambda = c/\nu \simeq 21.1\,cm$

Total Spin of Two Spin-1/2 Particles

$\hat{S}_x = \hat{S}_x^e + \hat{S}_x^p \quad \hat{S} = \hat{S}^e + \hat{S}^p, \quad \hat{S}^2 = (\hat{S}^e + \hat{S}^p)^2 = (\hat{S}^e)^2 + (\hat{S}^p)^2 + 2\hat{S}^e \cdot \hat{S}^p \quad [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$

$\hat{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle \quad \hat{J}_1 \cdot \hat{J}_2 = \frac{1}{2}(\hat{J}^2 - \hat{J}_1^2 - \hat{J}_2^2)$

$\hat{J}_z |j,m\rangle = m\hbar |j,m\rangle$

$\hat{J}_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)}\,\hbar\,|j,m+1\rangle$

$\hat{J}_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\,\hbar\,|j,m-1\rangle$

Coupling any Two Angular Momenta

Ex) Spin-1 & Spin-1/2

$|m_1, m_2\rangle = |+1, +1/2\rangle$ $\quad m_1 + m_2 = M = +3/2 \quad |J, M\rangle = |3/2, +3/2\rangle$

$|+1, -1/2\rangle, |0, +1/2\rangle \quad +1/2$

$|0, -1/2\rangle, |-1, +1/2\rangle \quad -1/2$

$|-1, -1/2\rangle \quad -3/2$

$$\hat{J}_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\,\hbar\,|j, m-1\rangle$$

$$\hat{J}_- |3/2, +3/2\rangle = (\hat{j}_{1-} + \hat{j}_{2-})|+1, +1/2\rangle$$