Phyics 110A Midterm Cheat Sheet $\nabla\cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

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Vector Fields

$$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\mathbf{r});\ \nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2};\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2};\ \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Electrostatics

$$\underline{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}};\ \underline{F} = Q\underline{E};\ E(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau',\ \oint_{\partial\mathcal{V}}\underline{E}\cdot d\underline{a} = \frac{1}{\epsilon_0}\int_{\mathcal{V}}\rho\,d\tau = \frac{Q_{enc}}{\epsilon_0} \Leftrightarrow \nabla\cdot\underline{E} = \frac{\rho}{\epsilon_0};\ \oint\underline{E}\cdot d\underline{l} = 0 \Leftrightarrow \nabla\times\underline{E}$$

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}}\underline{E}\cdot d\underline{l} \Leftrightarrow \underline{E} = -\nabla V;\ \nabla^2 V = -\frac{\rho}{\epsilon_0};\ V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'\ (\leftarrow \text{implicitly sets ref. pt. at } \infty),\ E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0},\ E^{\parallel}_{above} = E^{\parallel}_{below},\ \underline{E}_{ab.} - \underline{E}_{below} = \frac{\sigma}{\epsilon_0}$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0},\ \frac{\partial V}{\partial n} = \nabla V\cdot\hat{\underline{n}},\ W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \rightarrow W = \frac{1}{2}\int\rho V\,d\tau,\ W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\,d\tau$$

Conductors

(i) $\underline{E} = \underline{0}$ inside a conductor, (ii) $\rho = 0$ inside a conductor, (iii) Any net charge resides on the surface, (iv) A conductor is an equipotential

Capacitors

$C \equiv \frac{Q}{V}$, $C = \frac{A\epsilon_0}{d}$ (// plate), $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$

Uniqueness Theorems

1. The Solution to Laplaces Eqn. in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\partial\mathcal{V}$

1'. The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries are specified

2. In a volume $\mathcal{V}$ surrounded by conductors & containing a specified charge density ρ, $\mathbf{E}$ is uniquely determined if Q_i is given.

Image Charges

From the uniqueness theorems, if we can come up with image charges outside of $\mathcal{V}$, then that yields V.

Separation of Variables

Set $V = X(x)Y(y)Z(z)$ or $V = R(r)\Theta(\theta)$. For spherical coords, let $V = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$; $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2\ldots$

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty} r^n\left(A_n\sin(n\phi) + B_n\cos(n\phi)\right) + r^{-n}\left(C_n\sin(n\phi) + D_n\cos(n\phi)\right)$$

Multipole Expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta' + \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots\right] = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int(r')^n P_n(\cos\theta')\rho(\mathbf{r}')d\tau'$$

$$p \equiv \int\underline{r}'\rho(\mathbf{r}')d\tau' \Rightarrow V_{dip}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\frac{\underline{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} \Rightarrow \underline{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(p\cdot\hat{\underline{r}})\hat{\underline{r}} - p\right]$$

Dielectrics

$$\underline{p} = \int\underline{P}\,d\tau;\ \sigma_b = \underline{P}\cdot\hat{\underline{n}};\ \rho_b = -\nabla\cdot\underline{P};\ \rho = \rho_b + \rho_f;\ \underline{D} = \epsilon_0\underline{E} + \underline{P} \Rightarrow \nabla\cdot\underline{D} = \rho_f \Leftrightarrow \oint\underline{D}\cdot d\underline{a} = Q_{f\,enc};\ D^{\perp}_a - D^{\perp}_b = \sigma_f,\ \underline{D}^{\parallel}_a - \underline{D}^{\parallel}_b = \underline{P}^{\parallel}_a - \ldots$$

In the case of a linear dielectric,

$$P = \epsilon_0\chi_e\underline{E},\ \underline{D} = \epsilon_0(1+\chi_e)\underline{E} = \epsilon\underline{E};\ \epsilon_r \equiv \frac{\epsilon}{\epsilon_0}$$