Physics 137A Eqn. Sheet (Midterm 1)

Harrison Bachrach

Basics

For energy Eigenfunctions

$E = \hbar\omega$, $p = \hbar k$, $k = \frac{2\pi}{\lambda}$, $\hbar = 1.05 \cdot 10^{-34}\, J \cdot s$; ψ is continuous, ψ' is continuous if $V(x)$ is finite.

$P(a \le x \le b) = \int_{-\infty}^{\infty} \psi^* \psi \, dx$; If $\psi(x)$ is normalizable, then $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \text{const.}$ | If $V(-x) = V(x)$ then we can take ψ to be even or odd | $\sigma_x \sigma_p \ge \frac{\hbar}{2}$

$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*_{(x)} x \, \psi_{(x)} dx = \int_{-\infty}^{\infty} \phi^*(p) \left(i\hbar \frac{\partial}{\partial p}\right) \phi(p) \, dp$ | $\hat{x}\psi(x) = x\psi$; $\hat{x}\phi(p) = i\hbar \frac{\partial}{\partial p} \phi(p)$ | Ehrenfest's Theorem $\langle p \rangle = m \frac{d\langle x \rangle}{dt}$, $\frac{d\langle p \rangle}{dt} = \left\langle -\frac{dV}{dx} \right\rangle$

$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*_{(x)} \left(-i\hbar \frac{\partial}{\partial x}\right) \psi_{(x)} dx = \int_{-\infty}^{\infty} \phi^*(p)\, p\, \phi(p) \, dp$ | $\hat{p}\psi(x) = -i\hbar \frac{\partial}{\partial x} \psi(x)$; $\hat{p}\phi(p) = p\phi(p)$

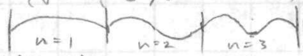
$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$; $\sigma_q^2 = \langle q^2 \rangle - \langle q \rangle^2$ | Must be normalized | $[\hat{x}, \hat{p}] = i\hbar$; $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

If $\hat{Q}\psi = \lambda\psi$: $\langle Q \rangle = \lambda$, $\sigma_Q = 0$; $P(q_0) = |\langle q_0 | \psi \rangle|^2 = \int_{-\infty}^{\infty} \psi^*_{q_0} \psi \, dx$ where $\hat{Q}\psi_{q_0} = q_0 \psi_{q_0}$ | For separable solutions: $\Psi(x,t) = \psi(x) e^{-i\frac{Et}{\hbar}}$

Infinite Square Well

$V(x) = \{0 \ (0 \le x \le a),\ \infty \ (\text{else})\}$: $\psi_n(x,t) = \left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) e^{i\hat{f}(t)} \ (0 \le x \le a),\ 0 \ (\text{else}) \right\}$ | $c_n = \int_{-\infty}^{\infty} \psi_n^* \Psi(x,0) \, dx$

$\hat{H}\psi_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \psi_n \Rightarrow E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2$



(Forms an orthonormal eigenbasis for energy) $\Rightarrow \sum_n |c_n|^2 = 1$ (Normalization Condition)

$\langle E \rangle = \sum |c_n|^2 E$

Free Particle

$V(x) = 0$; $\Psi_E(x,t) = \overbrace{A\exp(i(kx - \omega t))}^{\text{right-moving}} + \overbrace{B\exp(i(-kx - \omega t))}^{\text{left-moving}}$; $k = \frac{\sqrt{2mE}}{\hbar}$, $\omega = \frac{E}{\hbar}$

$\Psi_E(x,t)$ is a momentum & Energy eigenstate, but is non-physical & non-normalizable.

$\psi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0 x/\hbar} \Rightarrow \int_{-\infty}^{\infty} \psi^*_{p_0} \psi_p \, dx = \delta(p_0 - p)$; $\int_{-\infty}^{\infty} e^{\frac{ix\beta}{\hbar}} dx = 2\pi\hbar\, \delta(\beta)$

F.T. & R.F.T.

$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(\frac{ipx}{\hbar}\right) \phi(p) \, dp \Leftrightarrow \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(-\frac{ipx}{\hbar}\right) \psi(x) \, dx$

Qualitative Wavefunctions

CFR | CAR

(1) exponential decay $(E < V)$; oscillations $(E > V)$ (2) λ decreases when $(E - V)$ increases

(3) Amplitude increases as $(E - V)$ decreases. (4) $n-1$ nodes for n^{th} eigenstate $(n \ge 1)$

Finite Square Well

Bound State $(E < 0)$

$V(x) = \{-V_0 \ (-a \le x \le a);\ 0 \ (\text{else})\}$

By normalizability

$\text{TISE} \rightarrow \psi_I'' = \left(-\frac{2mE}{\hbar^2}\right)\psi_I = \kappa^2 \psi_I \Rightarrow \psi_I = Ae^{\kappa x} + \cancel{Be^{-\kappa x}}^{\,0}$; symmetrically, $\psi_{III} = \cancel{Fe^{\kappa x}}^{\,0} + Ge^{-\kappa x}$

$\text{TISE} \rightarrow \psi_{II}'' = -\frac{2m(E + V_0)}{\hbar^2} \psi_{II} = -k^2 \psi_{II} \Rightarrow \psi_{II} = C\sin(kx) + D\cos(kx)$; $\psi_E = \begin{cases} Ae^{\kappa x} & \text{I} \\ D\cos(kx) & \text{II (or) } C\sin(kx) \\ Ae^{-\kappa x} & \text{III (or) } -Ae^{-\kappa x} \end{cases}$

$D\cos(ka) = Ae^{-\kappa a}$

even: $-kD\sin(ka) = -\kappa A e^{-\kappa a} \Rightarrow \boxed{\kappa = k\tan(ka)} \rightarrow$ energy quantization: $z = ka$, $z_0 = \frac{a}{\hbar}\sqrt{2mV_0} \Rightarrow \tan z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$

Scattering State $(E > 0)$ — incoming, reflected, Transmitted

$\text{TISE} \rightarrow \psi_I = Ae^{ikx} + Be^{-ikx}$; $\psi_{III} = Fe^{ikx} + \cancel{Ge^{-ikx}}^{\,0}$; $\psi_{II} = C\sin(lx) + D\cos(lx)$; $k = \frac{\sqrt{2mE}}{\hbar}$, $l = \frac{\sqrt{2m(E + V_0)}}{\hbar}$

Probability Current $J = \frac{i\hbar}{2m}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^* \frac{\partial \psi}{\partial x}\right)$; if $\psi = Ae^{ipx/\hbar} \Rightarrow J = |A|^2 \frac{p}{m}$; $T \equiv \frac{J_{III}}{J_I^+}$; $R \equiv \frac{J_I^-}{J_I^+}$;

only if $V_{\text{incoming}} = V_{\text{outgoing}}$

$T = \frac{|F|^2}{|A|^2}$, $R = \frac{|B|^2}{|A|^2}$, $T + R = 1$

Delta Well Potential

Bound State

$V(x) = -\alpha\delta(x)$, $(\alpha > 0)$;

$\psi_I = Be^{-\kappa x}$, $\psi_{II} = Be^{\kappa x} \Rightarrow \psi = B'e^{-\kappa|x|}$; $\kappa = \frac{\sqrt{2m|E|}}{\hbar}$

$B = B'$ by continuity, $B = \sqrt{\kappa}$ by normalization; Integrate TISE $\rightarrow \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2} dx = -\frac{2m}{\hbar^2} \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} (E - V(x))\psi \, dx$

$\Rightarrow \frac{d\psi}{dx}\Big|_{-\epsilon}^{\epsilon} = -\frac{2m}{\hbar}\left[0 + \alpha\psi(0)\right] \Rightarrow -2\kappa = -\frac{2m}{\hbar^2}\alpha\sqrt{\kappa} \Rightarrow \kappa = \frac{m\alpha}{\hbar}$; $E = -\frac{m\alpha^2}{2\hbar^2}$; $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar}x\right)$ ← only bound state.

Scattering State

$\text{TISE} \rightarrow \psi_I = Ae^{ikx} + Be^{-ikx}$, $\psi_{II} = Fe^{ikx} + \cancel{Ge^{-ikx}}^{\,0} \Rightarrow$ By continuity, we get $A = \frac{i\beta}{1 - i\beta}$, $F = \frac{1}{1 - i\beta}$, $\beta \equiv \frac{m\alpha}{\hbar^2 k}$

$k \equiv \frac{\sqrt{2mE}}{\hbar} \Rightarrow T = \frac{1}{1 + (m\alpha^2/2\hbar^2 E)}$, $R = \frac{1}{1 + (2\hbar^2 E/m\alpha^2)}$

Postulates

(1) The state of a quantum system is given by a complex, normalized wavefunction $\Psi(x,t)$

(2) Observable quantities are represented by linear operators acting on wave functions.

(3.1) A precise measurement of an observable $\hat{Q}$ will yield one & only one of the eigenvalues of $\hat{Q}$.

(3.2) Given a state $\Psi(x, t_0)$, the probability of measuring a given eigenvalue q_i is given by $P(q_i) = |\langle q_i | \Psi(x,t_0) \rangle| = \left|\int_{-\infty}^{\infty} \Psi_i^*(x,t_0)\, \Psi_i(x,t_0) \, dx\right|$ where $\Psi_i(x,t)$ is a normalized eigenvector corresponding to q_i

(3.3) After a measurement of an observable $\hat{Q}$ at observable q_i, the wave function "collapses" to the normalized eigenstate $\Psi_i(x,t)$

(4) The time evolution of a system is given by $i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \hat{H}\, \Psi(x,t)$

$\text{TDSE} \Rightarrow i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

$\text{TISE} \Rightarrow -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = \hat{H}\psi = E\psi$

Mathematical Reference

(Normalized) Gaussian Wave Packet: $\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4} \exp\left(-\frac{x^2}{4\sigma_x^2}\right)$

$\langle x \rangle \Rightarrow$ odd func! zero by sym. or use u-sub

$\langle x^2 \rangle \Rightarrow$ Use the following dirty trick:

$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = -\int_{-\infty}^{\infty} \frac{\partial}{\partial a} e^{-ax^2} dx$

$= -\frac{\partial}{\partial a} \int_{-\infty}^{\infty} e^{-ax^2} dx = -\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$

$\int_{-\infty}^{\infty} e^{-(ax^2 + bx)} dx \Rightarrow ax^2 + bx = ax^2 + 2\sqrt{a}\frac{b}{2\sqrt{a}}x + \frac{b^2}{4a} - \frac{b^2}{4a}$

$= \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a}$; let $u = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)$

$\rightarrow \int_{-\infty}^{\infty} \exp(\cdots) dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2 + \frac{b^2}{4a}} du = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$

$\int_0^{2\pi} \sin^2(x) dx = \pi$;

$\sin(u)\sin(v) = \frac{1}{2}(\cos(u - v) - \cos(u + v))$

$\cos(u)\cos(v) = \frac{1}{2}(\cos(u - v) + \cos(u + v))$

$\sin(u)\cos(v) = \frac{1}{2}(\sin(u + v) + \sin(u - v))$

$\sin(u) + \sin(v) = 2\sin\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right)$