Physics 137B: Midterm 2 Study Guide

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A set V is a vector space iff: ① $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$ ("ket."); ② $|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$; ③ $|\phi\rangle \in V$ s.t. $|\alpha\rangle + |\phi\rangle = |\alpha\rangle$;

④ $\forall |\alpha\rangle \in V, \exists |-\alpha\rangle$ s.t. $|\alpha\rangle + |-\alpha\rangle = |\phi\rangle$; ⑤ $c(|\alpha\rangle + |\beta\rangle) = c|\alpha\rangle + c|\beta\rangle$ and $(c+d)|\alpha\rangle = c|\alpha\rangle + d|\alpha\rangle$; ⑥ $c(d|\alpha\rangle) = (cd)|\alpha\rangle$; "bra"

⑦ $0|\alpha\rangle = |\phi\rangle$; ⑧ $1|\alpha\rangle = |\alpha\rangle$. Given a vector space V, the dual space $V^* \equiv \{f : V \to \mathbb{C} \mid f \text{ is linear}\}$. $\Rightarrow \langle\alpha| : \mathcal{H} \to \mathbb{C}$.

The Inner Product:

$I : V \times V \to \mathbb{C}$; $I(|\alpha\rangle, |\beta\rangle) = (I(|\beta\rangle, |\alpha\rangle))^*$; Pos. Definite: $I(|\alpha\rangle, |\alpha\rangle) \geq 0 \ \forall |\alpha\rangle$ & $I(|\alpha\rangle, |\alpha\rangle) = 0 \Leftrightarrow |\alpha\rangle = |\phi\rangle$; $\langle\alpha| \equiv I(\langle\alpha|, \cdot) \Rightarrow \langle\alpha|\beta\rangle \equiv I(|\alpha\rangle, |\beta\rangle)$

Linearity in 2nd slot: $I(|\alpha\rangle, b|\beta\rangle + c|\gamma\rangle) = bI(|\alpha\rangle, |\beta\rangle) + cI(|\alpha\rangle, |\gamma\rangle)$; Anti-Linearity in 1st slot: $I(a|\alpha\rangle + b|\beta\rangle, |\gamma\rangle) = a^* I(|\alpha\rangle, |\gamma\rangle) + b^* I(|\beta\rangle, |\gamma\rangle)$; $\| |\alpha\rangle \| = \sqrt{\langle\alpha|\alpha\rangle}$; Schwartz inequality: $|\langle\alpha|\beta\rangle|^2 \leq \|\alpha\|^2 \|\beta\|^2$

orthonormal basis: $\langle e^i | e_j \rangle = \delta_{ij}$; $\boxed{\text{Operators}}$ $\hat{Q} : V \to V$; $\hat{Q}(a|\alpha\rangle + b|\beta\rangle) = a\hat{Q}|\alpha\rangle + b\hat{Q}|\beta\rangle$; $(\langle\alpha|\hat{Q})|\beta\rangle = \langle\alpha|(\hat{Q}|\beta\rangle)$; $\langle Q \rangle = \langle\psi|\hat{Q}|\psi\rangle$; $\hat{1} = \sum_i |e_i\rangle\langle e^i|$

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$\underline{w}^\dagger \underline{v} = w_i v^i$; $(\underline{A}\,\underline{v})^i = A^i_j v^j$; $(\underline{w}^\dagger \underline{A}) = w^j A_i^{\ j}$; $(\underline{A}\,\underline{B})^i_j = A^i_k B^k_j$; $(\underline{v}\,\underline{w}^\dagger)^i_j = v^i w_j$; $|\psi\rangle = \psi^i |e_i\rangle$, $\langle\phi| = \phi_i \langle e^i|$, $\psi_i = (\psi^i)^*$; $\langle\alpha| = (|\alpha\rangle)^\dagger$

$\hat{Q} = Q^i_j |e_i\rangle\langle e^j| \Leftrightarrow Q^i_j = \langle e^i|\hat{Q}|e_j\rangle \Leftrightarrow Q = \begin{pmatrix} Q^1_1 & \cdots & Q^1_N \\ \vdots & & \vdots \\ Q^N_1 & \cdots & Q^N_N \end{pmatrix}$; $I(\hat{Q}^\dagger|\alpha\rangle, |\beta\rangle) = I(|\alpha\rangle, \hat{Q}|\beta\rangle)$; Hermetian: $\hat{Q}^\dagger = \hat{Q}$, Anti-Hermetian: $\hat{Q}^\dagger = -\hat{Q}$

Any operator is the sum of a Hermetian & Anti-Hermetian Part, If $\hat{Q}$ is Hermetian, $i\hat{Q}$ is anti-Hermetian, Unitary: $\hat{Q}^\dagger = \hat{Q}^{-1}$. If $\hat{Q}$ is Hermetian:

$\Rightarrow$ All its eigenvalues are real, eigenvalues belonging to distinct eigenvectors are orthogonal, & its eigenvec. span the Hilbert space.

$\Rightarrow \hat{Q} = \sum_i \lambda_i |g^i\rangle\langle g_i|$. $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

Continuous Bases:

$\hat{x}|x\rangle = x|x\rangle$; $\hat{p}|p\rangle = p|p\rangle$; $\langle x|x'\rangle = \delta(x - x')$; $\langle p|p'\rangle = \delta(p - p')$; $\hat{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$; $\langle x|\alpha\rangle = \psi_\alpha(x)$; $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

Changing Bases:

$\hat{U}|f_i\rangle = |e_i\rangle$; $U^i_j = \langle f^i|e_j\rangle \Leftrightarrow |e_j\rangle = U^i_j |f_i\rangle$; $\hat{U}^\dagger = \hat{U}^{-1}$; $|f_i\rangle = \hat{U}^\dagger|e_i\rangle$; $\Psi_{[f]} = U^\dagger \Psi_{[e]}$; $Q_{[e]} = U Q_{[f]} U^\dagger$; $U_{[e]} = U_{[f]}$ Unitarity & Hermeticity ind. of basis.

Postulate 1: A given quantum system is desribed by a Hilbert space. The state of a system at any given time is given by the normalized vec. in the Hilbert space.

Postulate 2: Observable quantities of a system are given by Hermetian operators on the Hilbert space of the system.

Postulate 3: The only possible results of measuring an observable are the eigenvalues of the corresponding eigenvector.
$\hat{P}_{\lambda_i} = |g_{\lambda_i}\rangle\langle g_{\lambda_i}| \Rightarrow P(\lambda_i) = \|\hat{P}_{\lambda_i}|\psi\rangle\| = \langle\psi|\hat{P}_\lambda|\psi\rangle = |\langle g_{\lambda_i}|\psi\rangle|^2$ if $\sigma_Q = 0$ (Recall $\sigma_Q^2 = \langle Q^2\rangle - \langle Q\rangle^2$) then $\hat{Q}|\psi\rangle = \langle Q\rangle|\psi\rangle$

Postulate 4: The dynamics of a system are given by the Hamiltonian operator $\hat{H}$ via $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$

Postulate 5: Immediately after a measurement, the state collapses. The new state is the normalized projection onto the rel. eig. sp.

$$|\psi(T^+)\rangle = \frac{\hat{P}_\lambda|\psi(T^-)\rangle}{\sqrt{\langle\psi(T^-)|\hat{P}_\lambda|\psi(T^-)\rangle}}$$

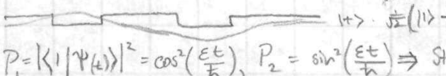
Two-State System

Two Observables:
$\hat{H} \Rightarrow$ Energy; $\hat{W} \Rightarrow$ "Which Well"

$|1\rangle \Rightarrow \hat{W}|1\rangle = 1|1\rangle$
$|2\rangle \quad \hat{W}|2\rangle = 2|2\rangle$; $|\psi\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle \Rightarrow \langle W\rangle = |c_1|^2 + 2|c_2|^2$. $W = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Approx: If wells are far apart, $\hat{H}|1\rangle = E|1\rangle$, $\hat{H}|2\rangle = E|2\rangle \Rightarrow$ Degenerate spectrum. As we bring them closer together,

$\hat{H}|1\rangle = E|1\rangle - \varepsilon|2\rangle$, $\hat{H}|2\rangle = E|2\rangle - \varepsilon|1\rangle \Rightarrow \hat{H} = E(|1\rangle\langle 1| + |2\rangle\langle 2|) - \varepsilon(|1\rangle\langle 2| + |2\rangle\langle 1|) \Rightarrow H = \begin{pmatrix} E & -\varepsilon \\ -\varepsilon & E \end{pmatrix} \to$ energies: $E \pm \varepsilon \Rightarrow$ Corresponding eigenstates $|+\rangle, |-\rangle$



$|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$
$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ $\Rightarrow P(E \pm \varepsilon) = \frac{1}{2} \pm \mathrm{Re}(c_1^* c_2)$; $|\psi(0)\rangle = |1\rangle \Rightarrow |\psi(t)\rangle = e^{-iEt/\hbar}\left(\cos\left(\frac{\varepsilon t}{\hbar}\right)|1\rangle + i\sin\left(\frac{\varepsilon t}{\hbar}\right)|2\rangle\right) \Rightarrow P_\pm = \frac{1}{2}$

$P_1 = |\langle 1|\psi(t)\rangle|^2 = \cos^2\left(\frac{\varepsilon t}{\hbar}\right)$, $P_2 = \sin^2\left(\frac{\varepsilon t}{\hbar}\right) \Rightarrow$ States of definite energy are not definite well-ness $\Leftrightarrow [\hat{H}, \hat{W}] \neq 0$. $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$

Simple Harmonic Oscillator

$V(x) = \frac{1}{2}m\omega^2 x^2$; $\hat{H} = \frac{1}{2m}((m\omega\hat{x})^2 + \hat{p}^2) = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2}) = \hbar\omega(\hat{a}_-\hat{a}_+ - \frac{1}{2})$; $\hat{a}_+ = \frac{1}{\sqrt{2\hbar\omega m}}[-i\hat{p} + m\omega\hat{x}]$, $\hat{a}_- = \frac{1}{\sqrt{2\hbar\omega m}}[+i\hat{p} + m\omega\hat{x}] \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$, $\hat{p} = i\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ - \hat{a}_-)$

$\hat{a}_+^\dagger = \hat{a}_-$, $\hat{a}_-^\dagger = \hat{a}_+$, $[\hat{a}_+, \hat{a}_-] = 1$; $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|0\rangle$, $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$, $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$; $\mathbb{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$. $\hat{N} \equiv \hat{a}_+\hat{a}_-$; $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$. $\hat{N}|n\rangle = n|n\rangle$

$(\mathbb{P})^n_{n'} = i\sqrt{\frac{\hbar\omega m}{2}}\left(\sqrt{n'+1}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n,n'-1}\right)$

OR:

$-\frac{\hbar}{2m}\psi'' + \frac{1}{2}m\omega^2 x^2\psi = E\psi$; $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, $K = \frac{2E}{\hbar\omega} \Rightarrow \psi''(\xi) = (\xi^2 - K)\psi(\xi)$. Ansatz: $\psi(\xi) = h(\xi)e^{-\xi^2/2}$. $\Rightarrow h'' - 2\xi h' + (K-1)h = 0$; Ansatz: $h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$

Plugging into the $h(\xi)$ diff eq $\Rightarrow a_{j+2} = \frac{2j + 1 - K}{(j+1)(j+2)}a_j \Rightarrow$ Physical solutions $\Leftrightarrow$ series termination $\Rightarrow K = 2n+1 \Rightarrow E_n = \hbar\omega(n + \frac{1}{2})$

$\Rightarrow \psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$; $H_0 = 1$, $H_1 = 2\xi$, $H_2 = 4\xi^2 - 2$, $H_3 = 8\xi^3 - 12\xi$